Machine Learning Methods for Neural Data Analysis

Decoding neural spike trains

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STATS 220/320 (NBIO220, CS339N). Winter 2023.

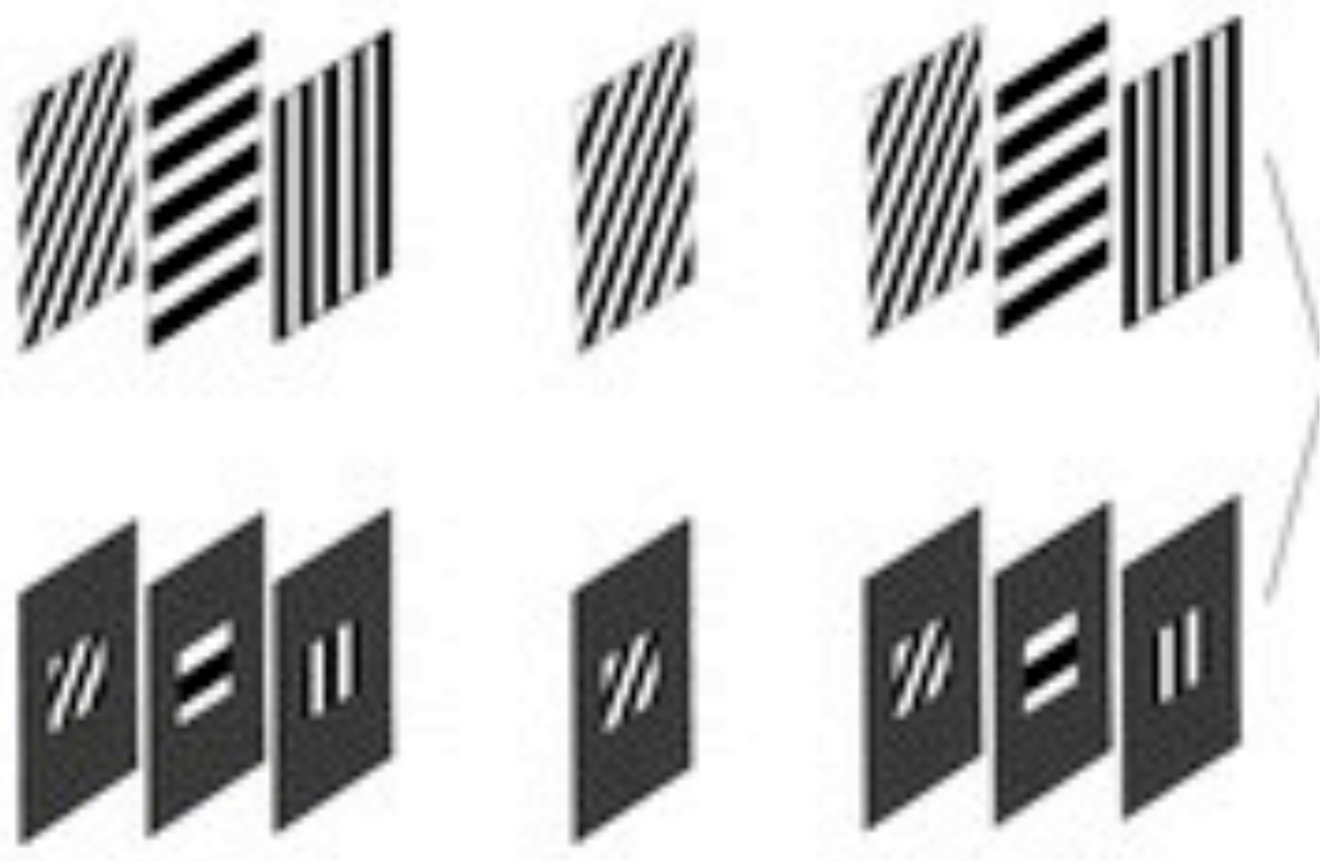
Agenda

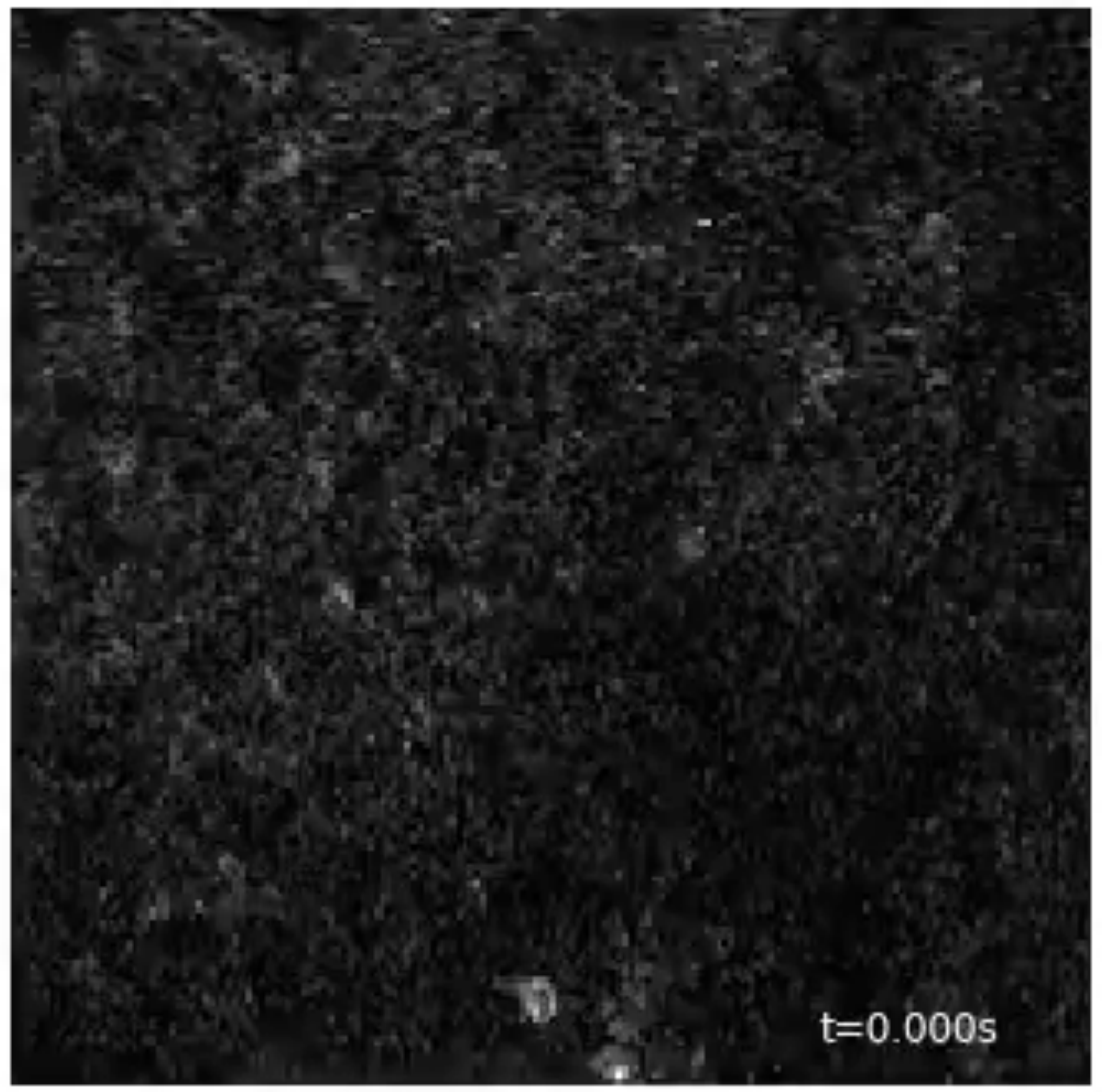
Decoding neural spike trains

- Bayesian decoders
  - A straw man model, just for illustration
  - An aside on the multivariate Gaussian distribution
  - Improving upon the basic model
- “Direct” decoders and structured prediction

Big picture

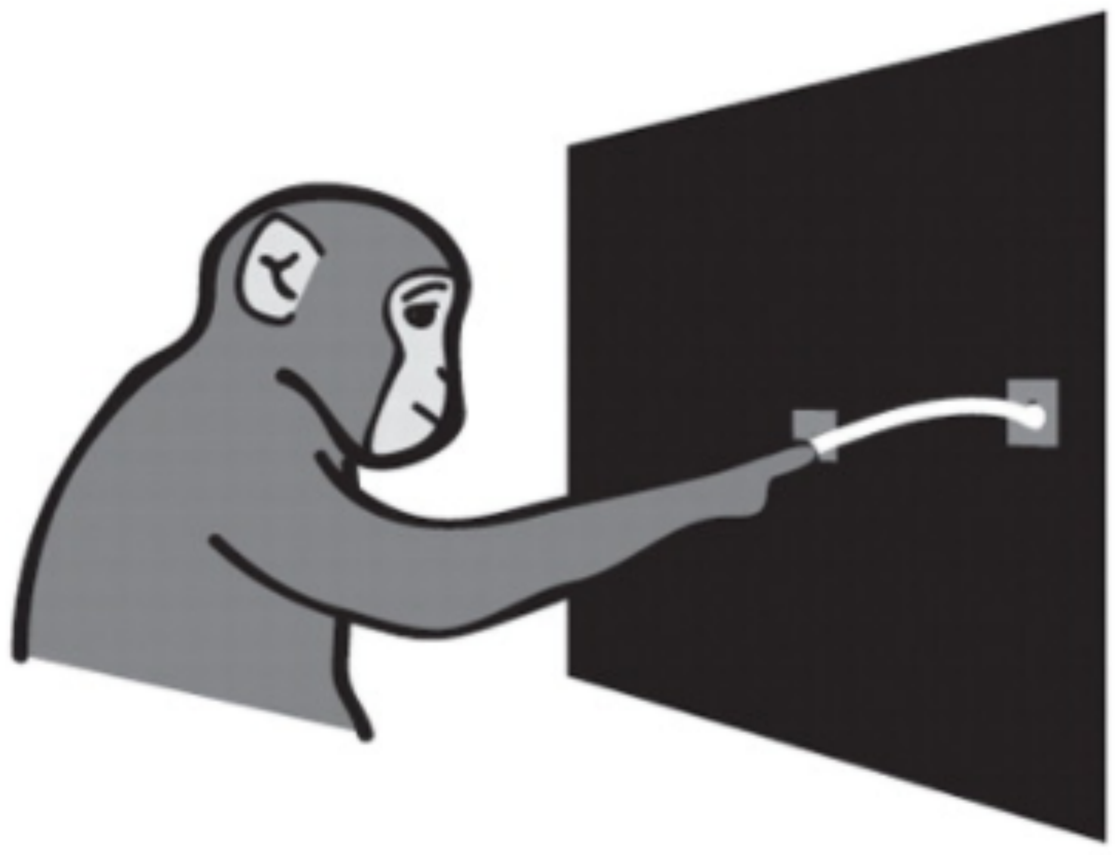
Stimulus —*encoding*→ Neural Recording —*decoding*→ Behavioral Video



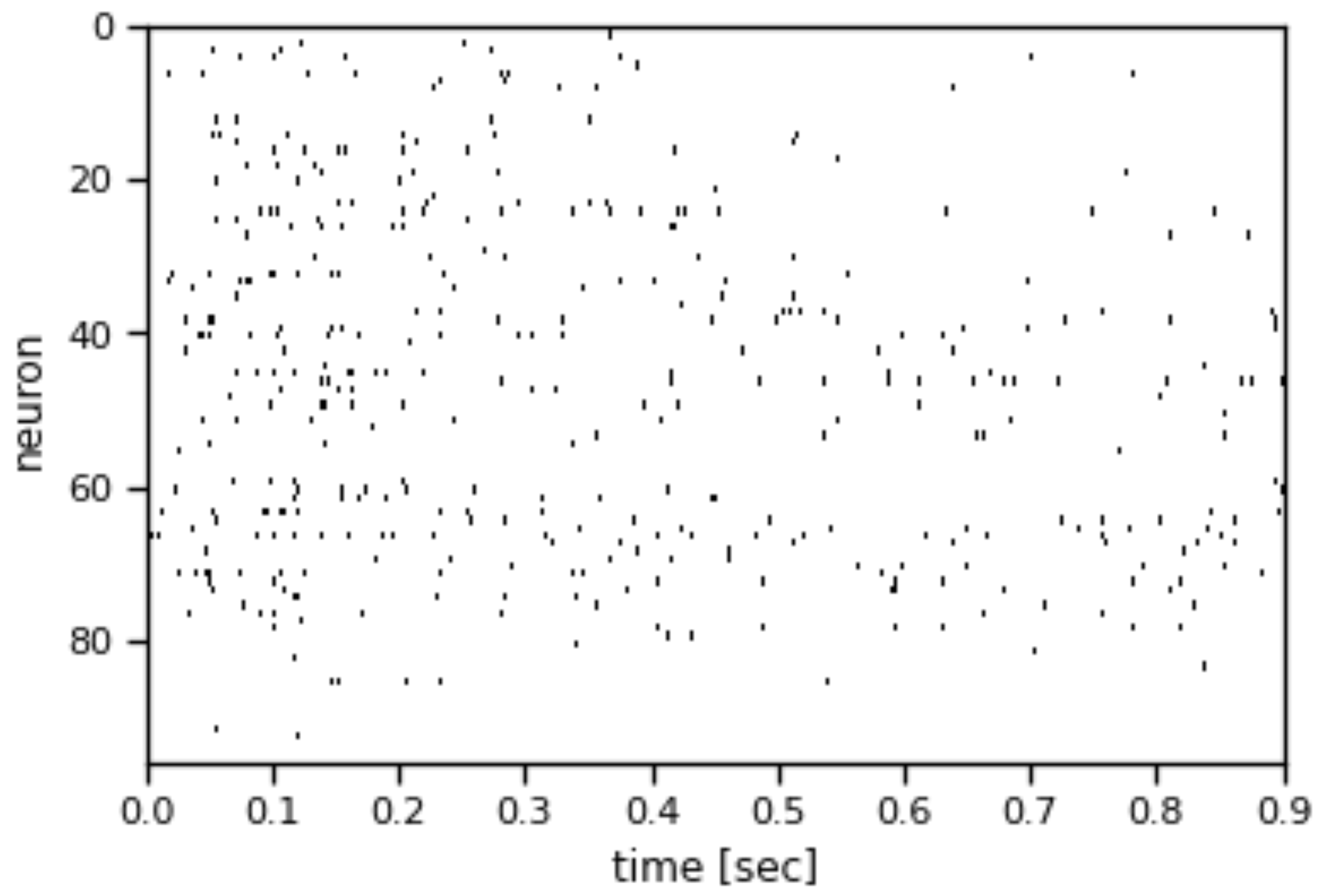


In statistics lingo, it's all regression.

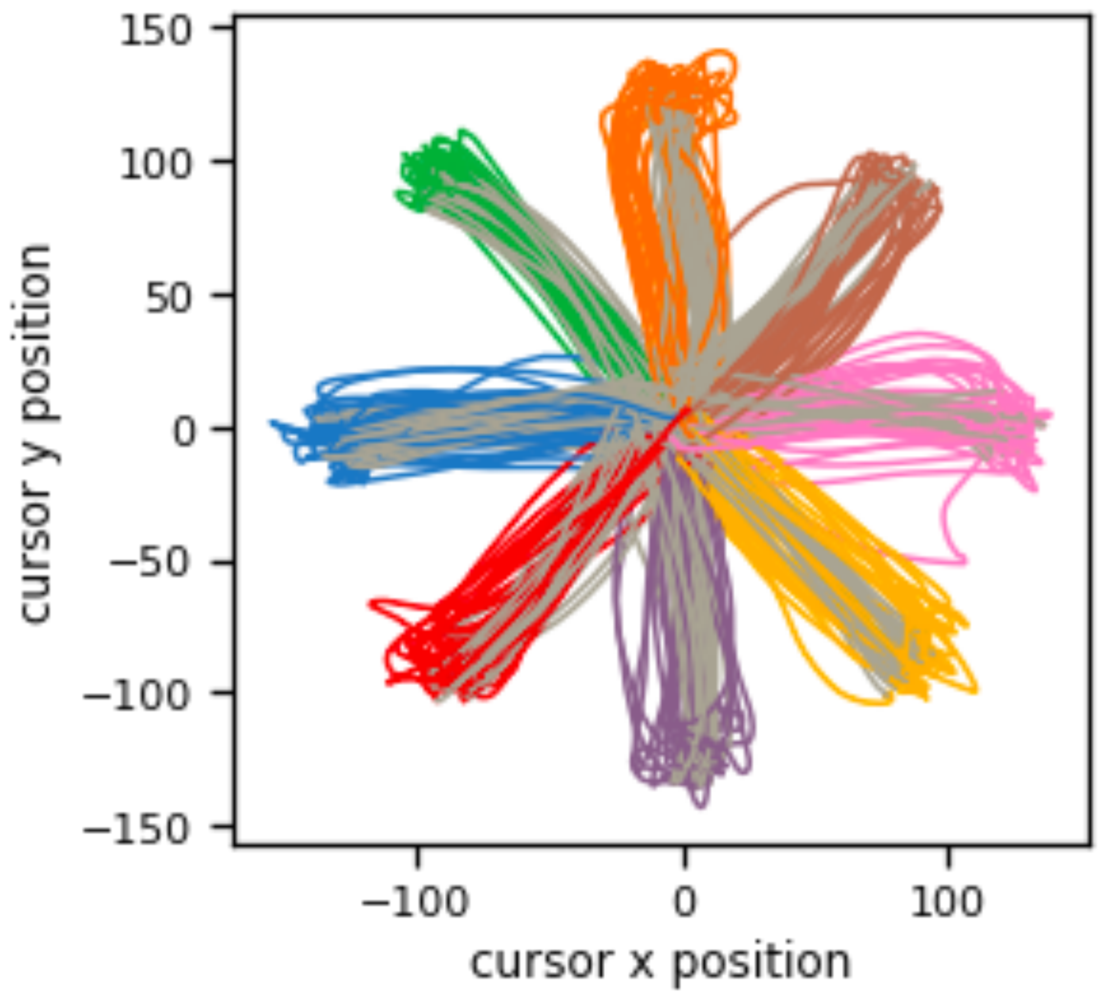
Decoding movement from recordings in motor cortex



Y



X



GOAL: estimate $p(X \mid Y)$

Shenoy Lab (Stanford

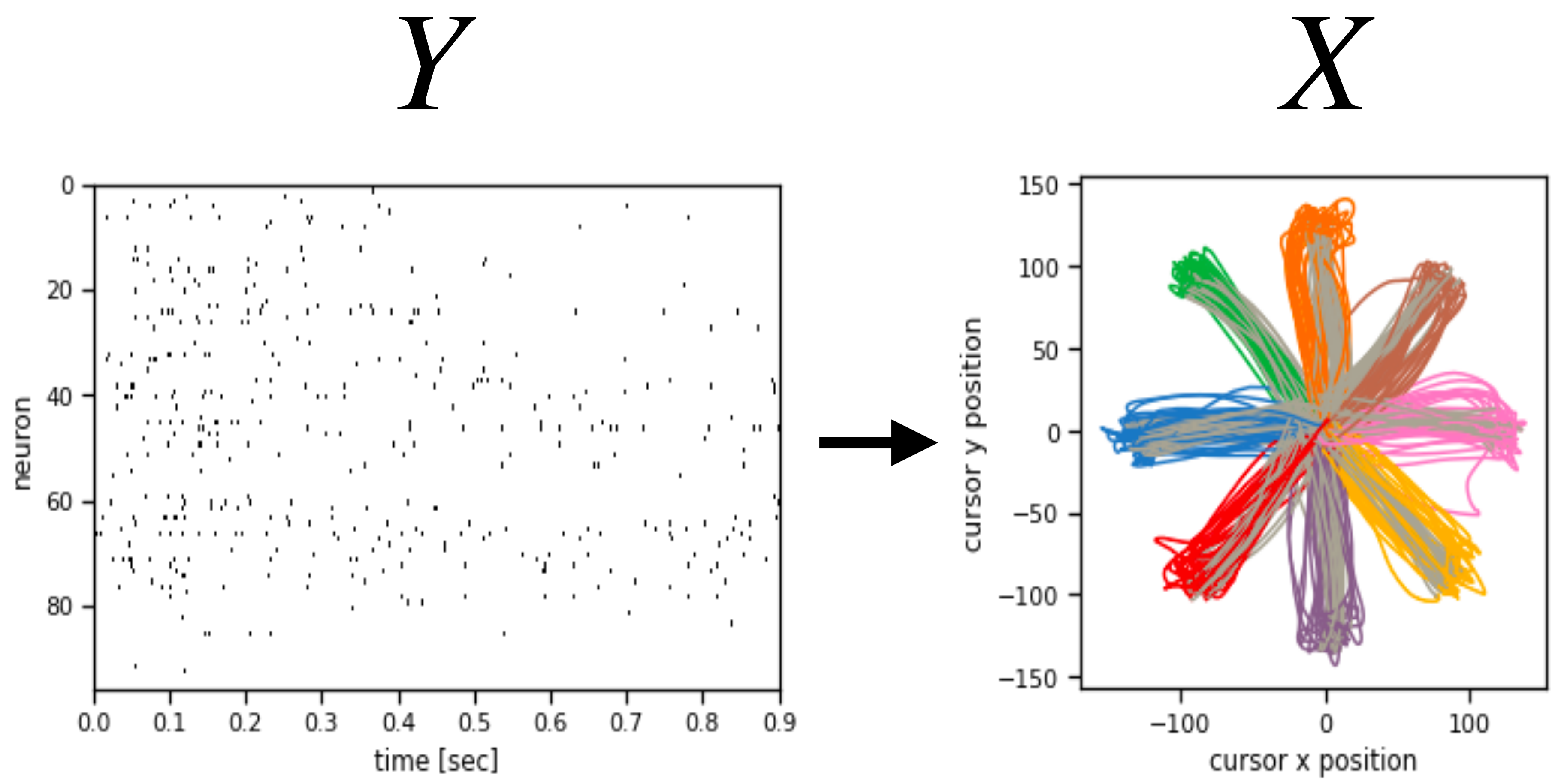
Krishna Shenoy, 1968-2023 | Photo by Rod Searcey

https://engineering.stanford.edu/magazine/krishna-shenoy-engineer-who-reimagined-how-brain-makes-body-move-dies-54

Decoding movement from neural spike trains

Brainstorming

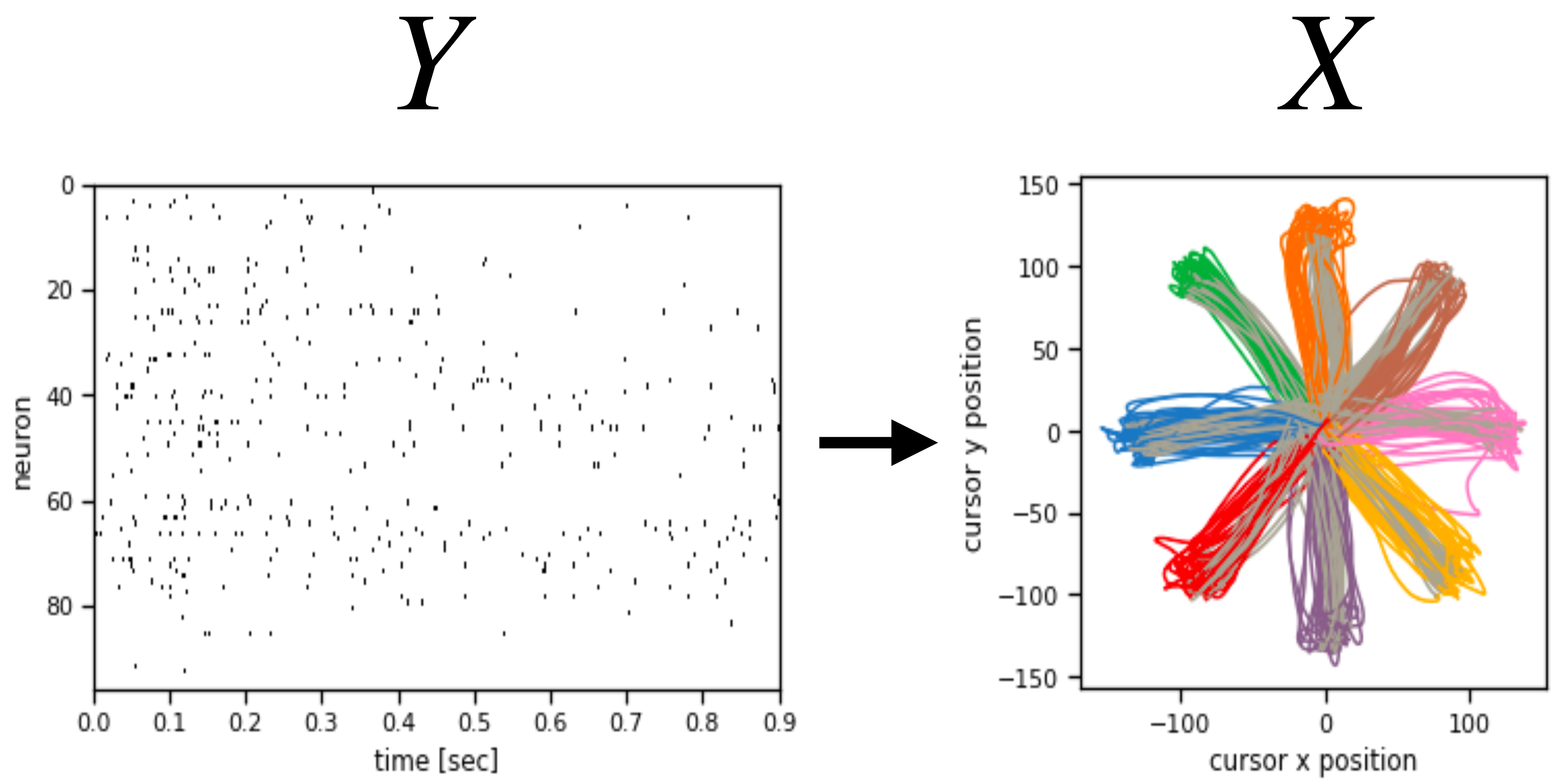
- How would you approach this problem?



Decoding movement from neural spike trains

Brainstorming

- It's just a regression problem… let's use the same techniques (GLMs, CNNs, etc) that we used for encoders.
    - I'll call these "direct" decoders, and we'll return to this idea in the second half of lecture.
- First, suppose we know something about the prior distribution of movement, $p(X)$. E.g. current position and velocity determine next position.
- Moreover, suppose we know something about what the neurons encode. E.g. suppose the neurons encode current velocity.
- Can we use that knowledge to inform our decoder?



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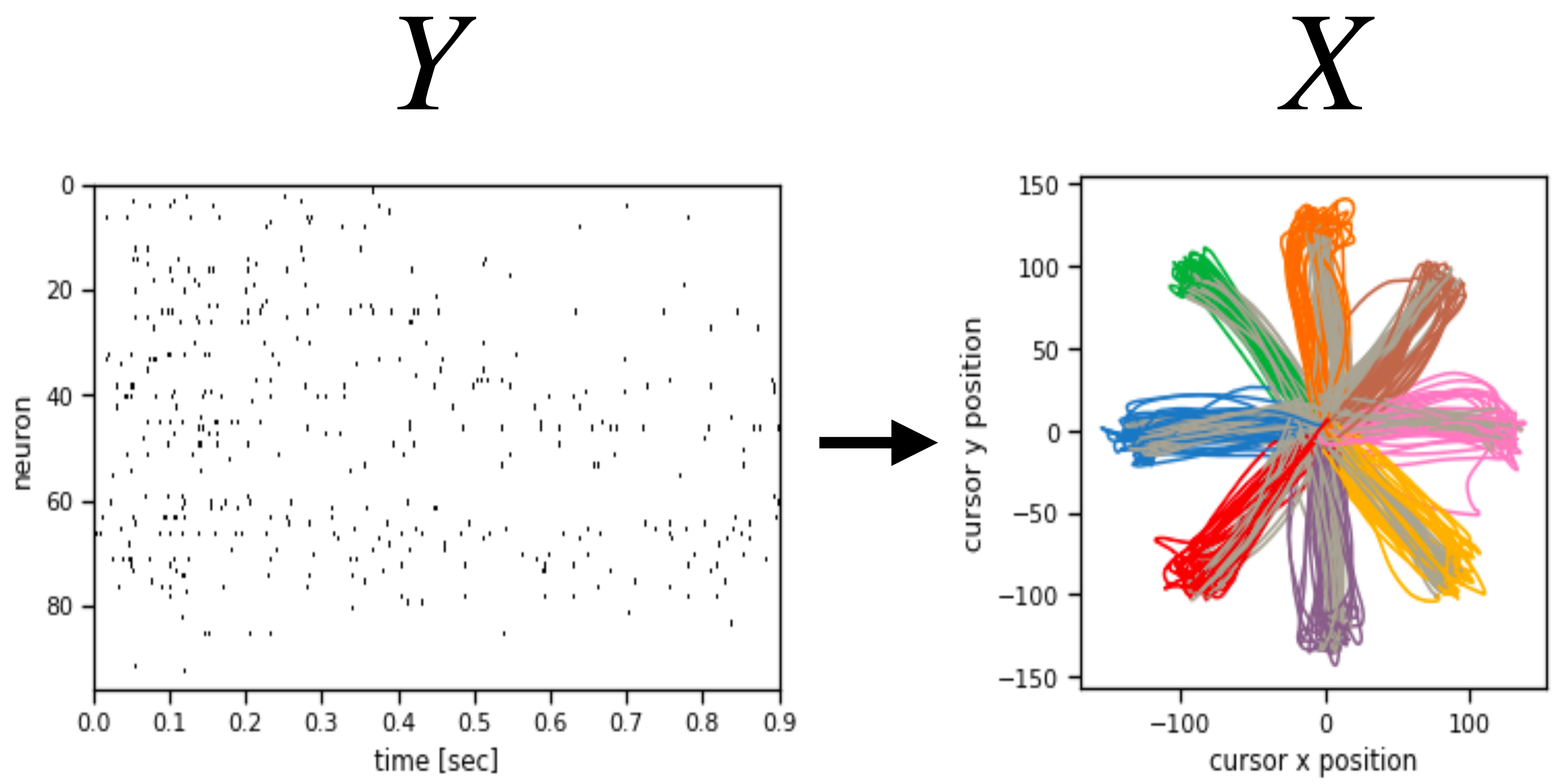
Bayesian decoders

- Bayes' Rule tells us how to combine a **prior** $p(X)$and a **likelihood** $p(Y \mid X)$ to obtain a **posterior**,

$$p(X \mid Y) = \frac{p(Y \mid X)p(X)}{p(Y)}$$
$$\propto p(Y \mid X)p(X)$$

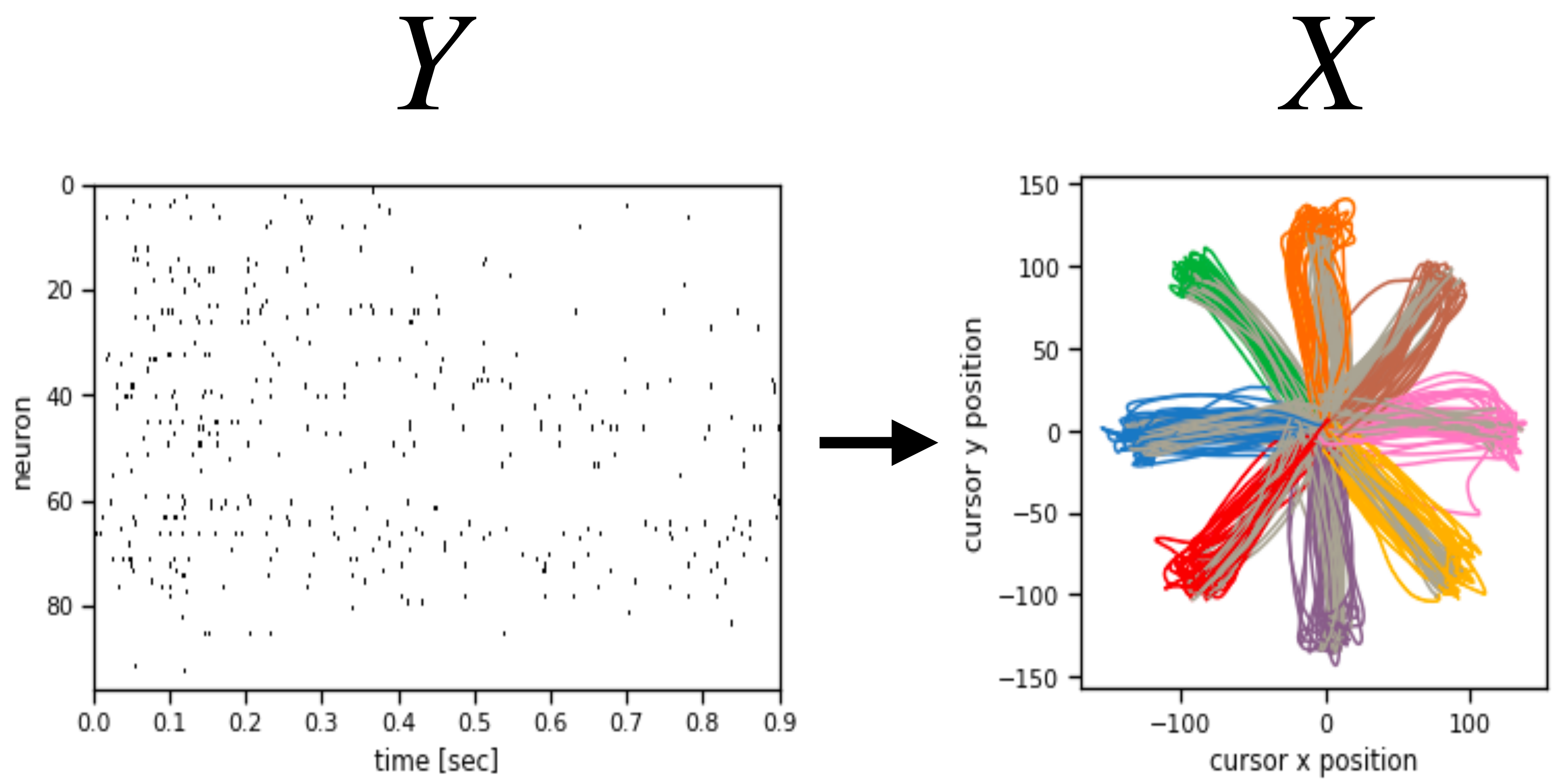
- Here, the likelihood is the **encoder** and the posterior is the **decoder**.



Decoding movement from neural spike trains

A very simple model

- Let $y_t \in \mathbb{N}^N$ denote the spike counts of N neurons at time t.
- Let $x_t \in \mathbb{R}^2$ denote the position of the cursor at time t.



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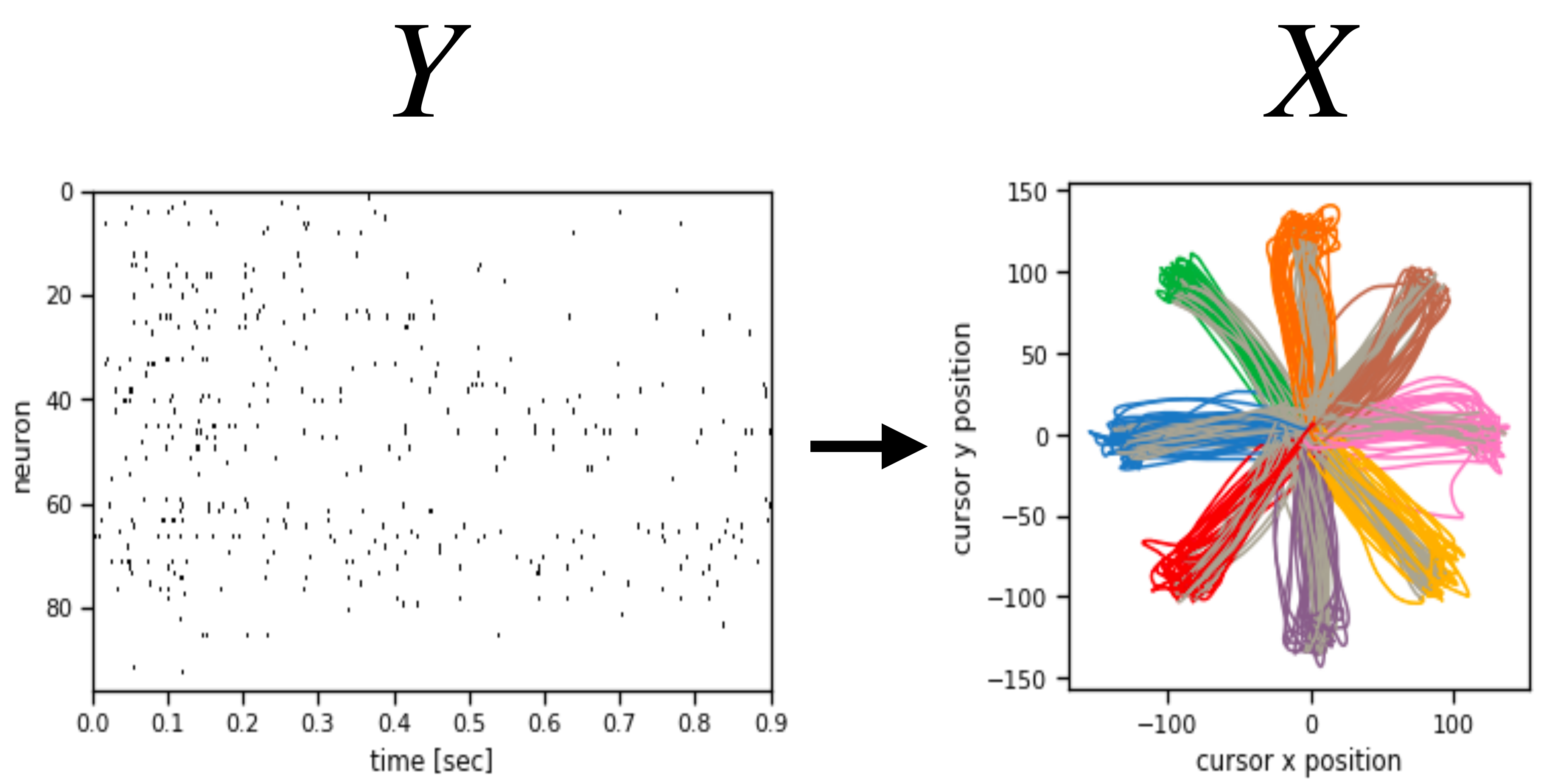
A simple example

Consider the following likelihood (i.e. encoder)…



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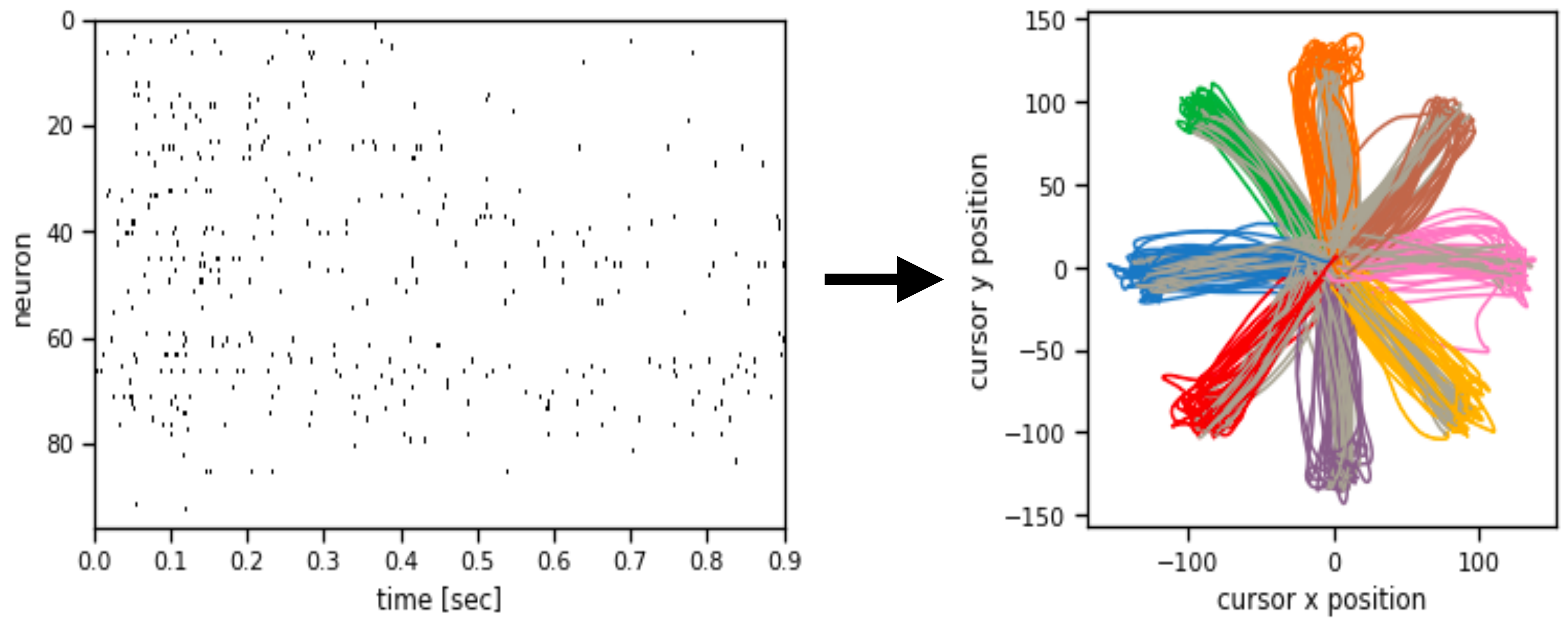
A simple example

- Consider the following prior…

Y X



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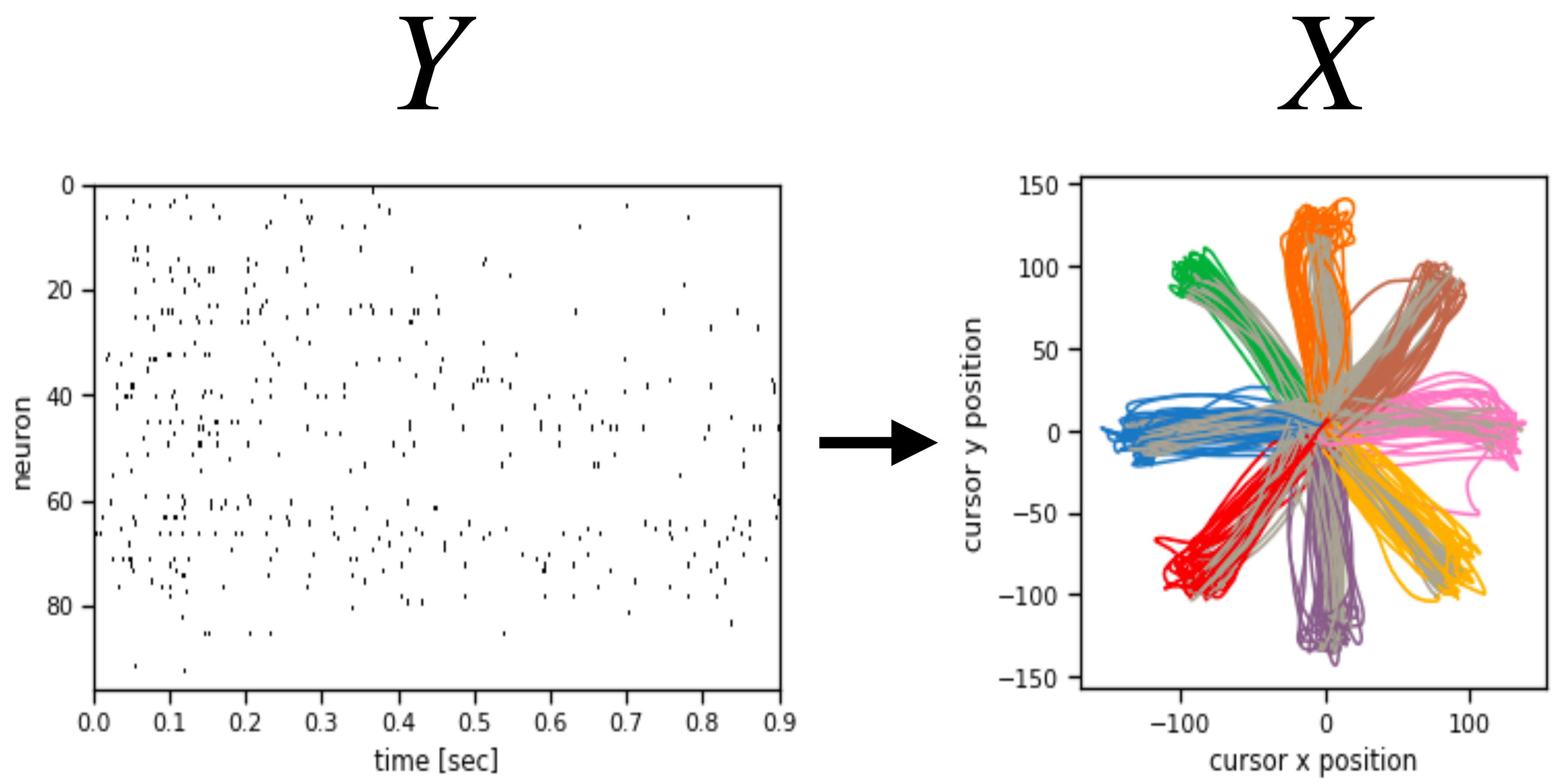
A simple example

Question: What's wrong with this model?



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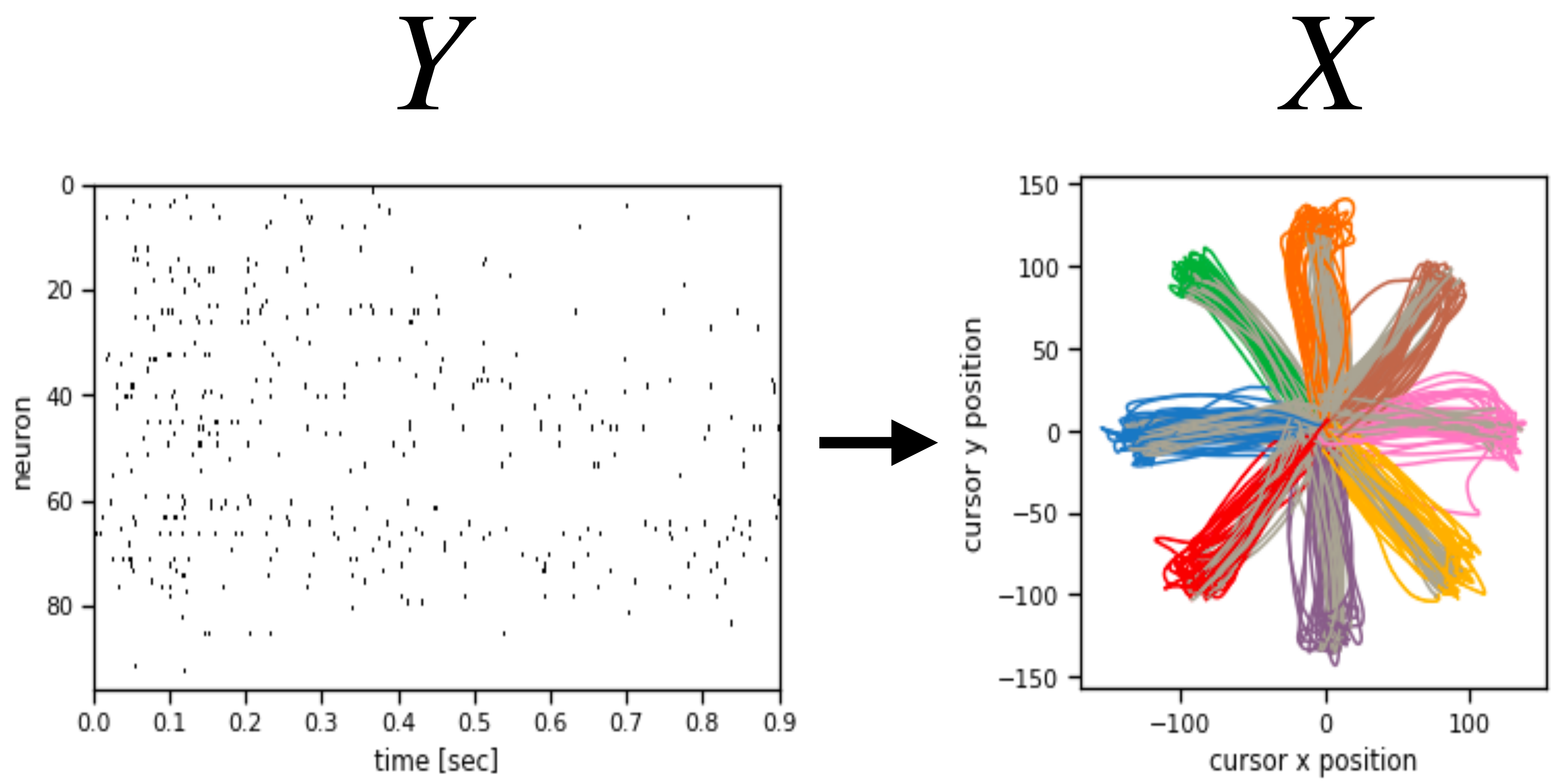
A simple example

Question: What's wrong with this model?

- Independent positions across time
- Gaussian model on counts?
- Condtionally independent counts
- Expected spike count is linear in x_t

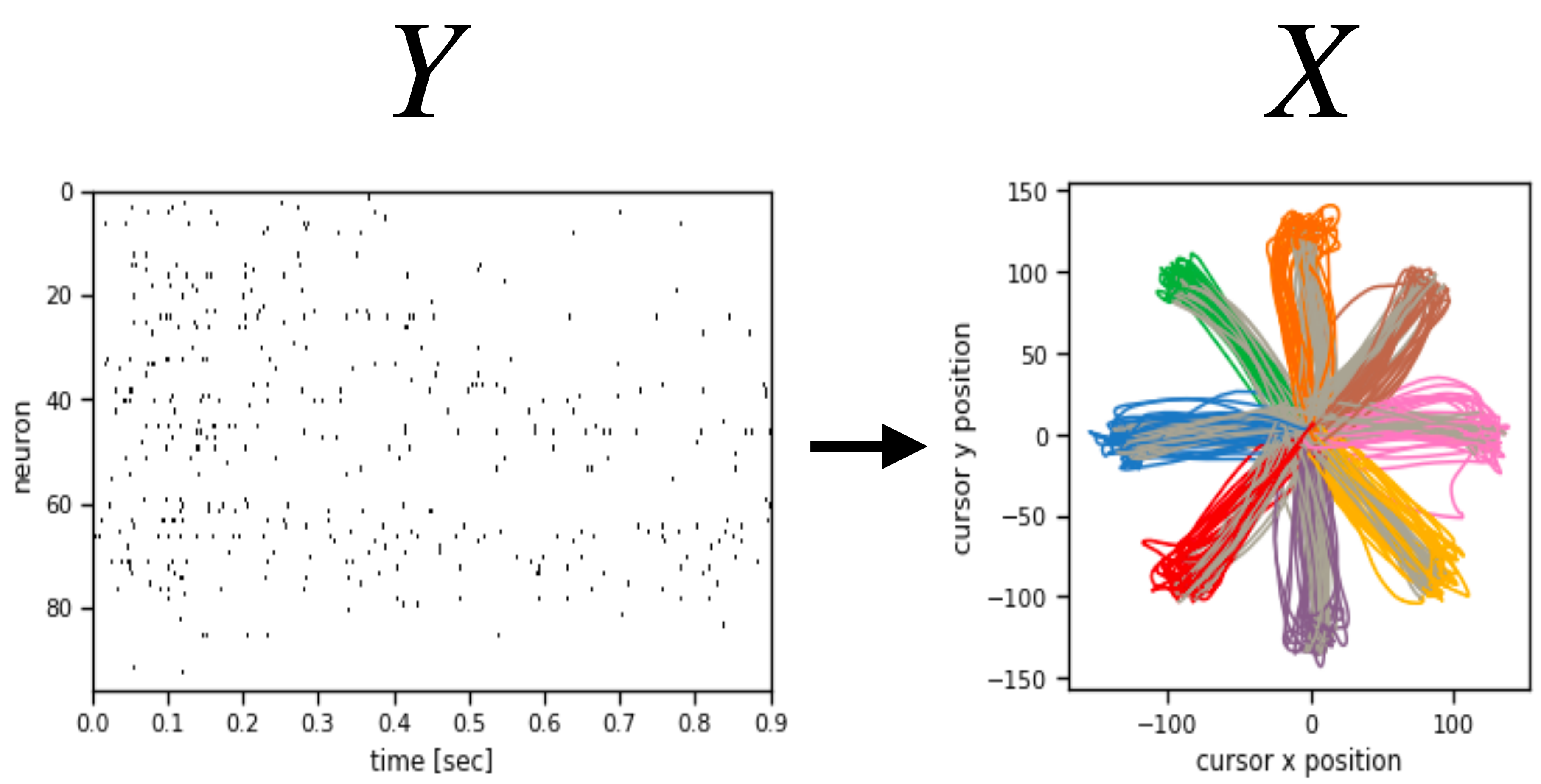


Decoding movement from neural spike trains

Deriving the posterior (decoder)

The one good thing about this model is it's easy to work with!

Derive the posterior…



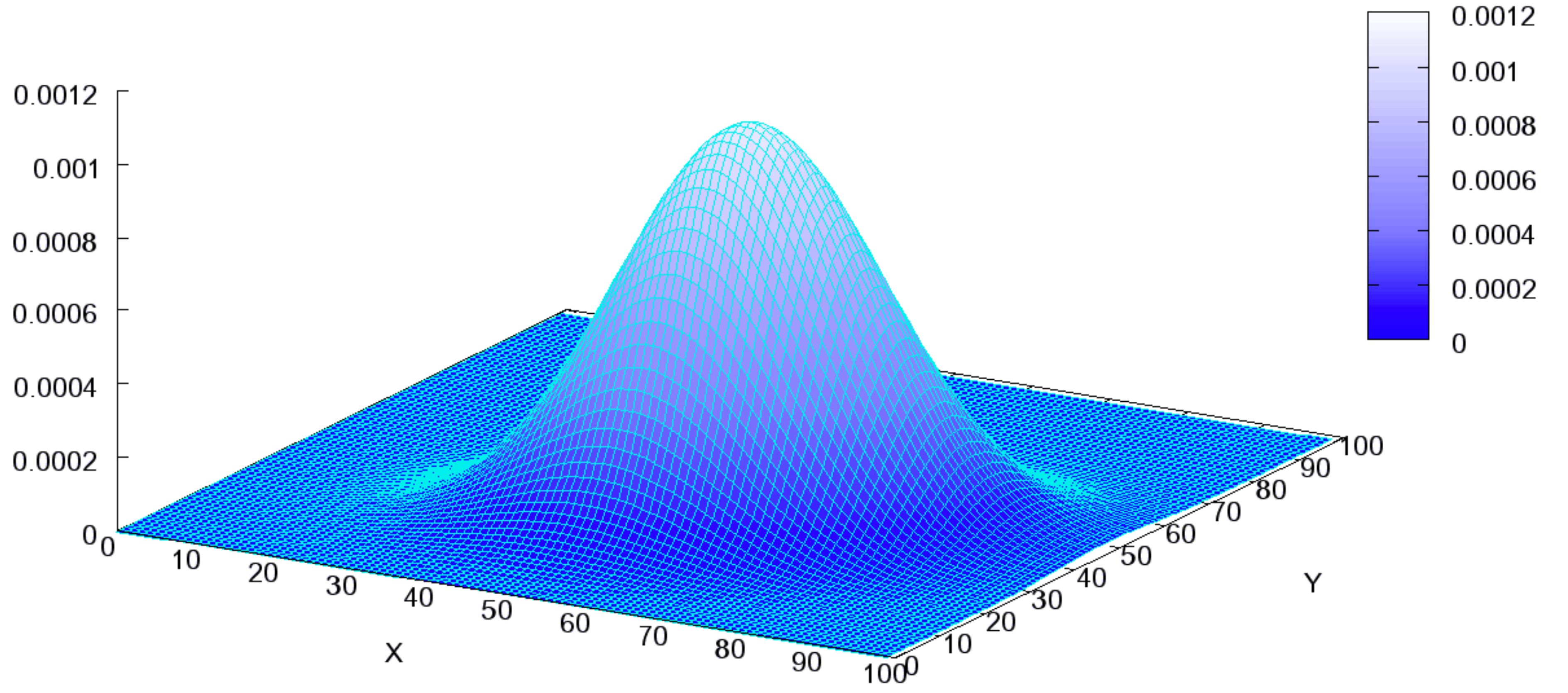
Aside: the multivariate Gaussian distribution

The multivariate Gaussian distribution

- Start with the standard normal distribution,
  - $z_d \sim \mathcal{N}(0,1) \iff p(z_d) = (2\pi)^{-1/2} \exp\left\{-\frac{z_d^2}{2}\right\}$
- Let $z = (z_1, \ldots, z_D)$ denote a vector of iid standard normal r.v.'s. Then,

$$
\begin{aligned}
p(z) &= \prod_{d=1}^{D} p(z_d) \\
&= \prod_{d=1}^{D} (2\pi)^{-1/2} \exp\left\{-\frac{z_d^2}{2}\right\} \\
&= (2\pi)^{-D/2} \exp\left\{-\frac{1}{2} z^\top z\right\}
\end{aligned}
$$

- We say $z \sim \mathcal{N}(0, I)$, a **multivariate normal distribution** with mean 0 and covariance I.

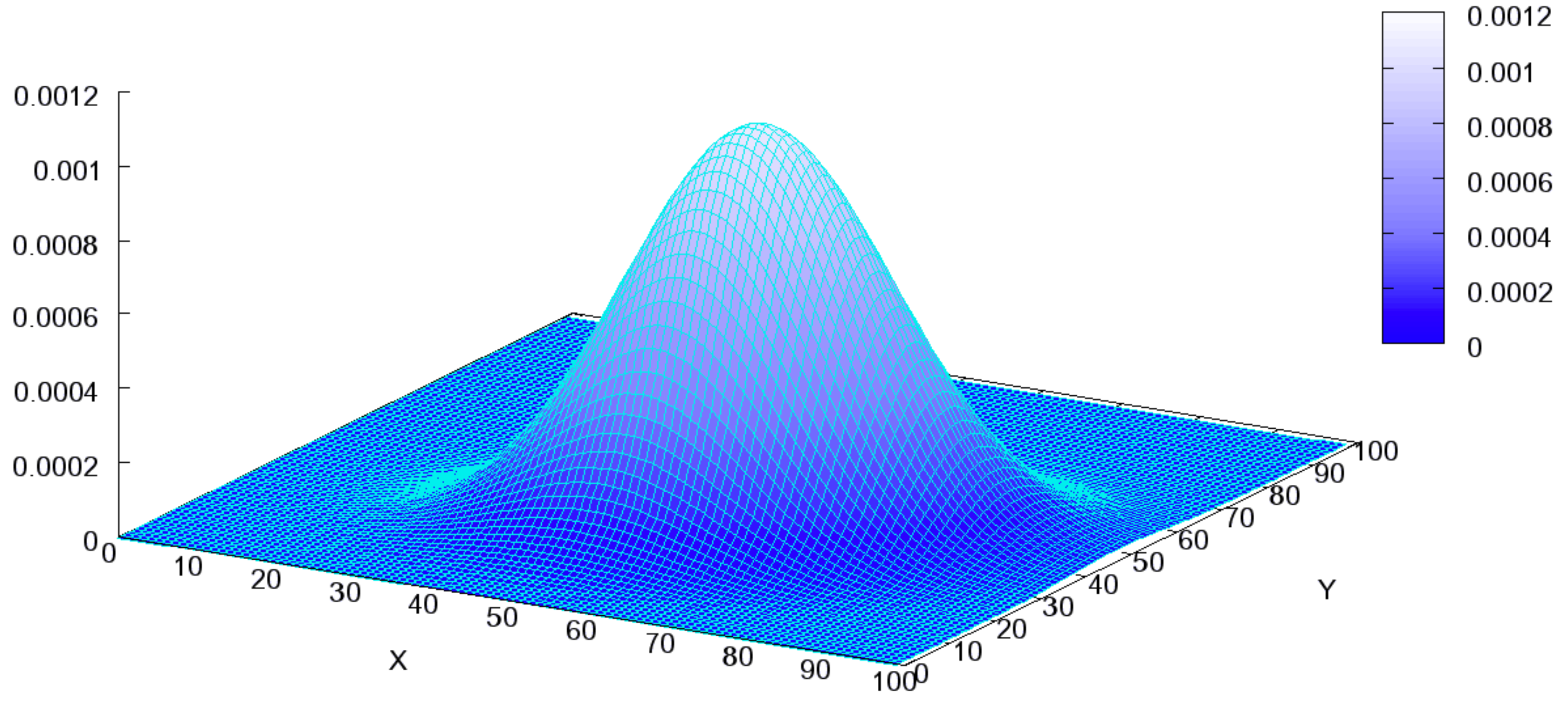


https://en.wikipedia.org/wiki/Multivariate_normal_distribution

Aside: the multivariate Gaussian distribution

- Now let $x = \mu + \Sigma^{1/2} z$ for $\mu \in \mathbb{R}^D$ and (invertible) $\Sigma^{1/2} \in \mathbb{R}^{D \times D}$.
- Then $z = \Sigma^{-1/2}(x - \mu)$.
- Change of variables formula:

$$
\begin{aligned}
p(x) &= \left| \frac{dz}{dx} \right| p(z(x)) \\
&= |\Sigma^{-1/2}| \mathcal{N}(\Sigma^{-1/2}(x - \mu), I) \\
&= (2\pi)^{-D/2} |\Sigma|^{-1/2} \exp\left\{ -\frac{1}{2}(x - \mu)^\top \Sigma^{-1}(x - \mu) \right\} \\
&\triangleq \mathcal{N}(x \mid \mu, \Sigma)
\end{aligned}
$$

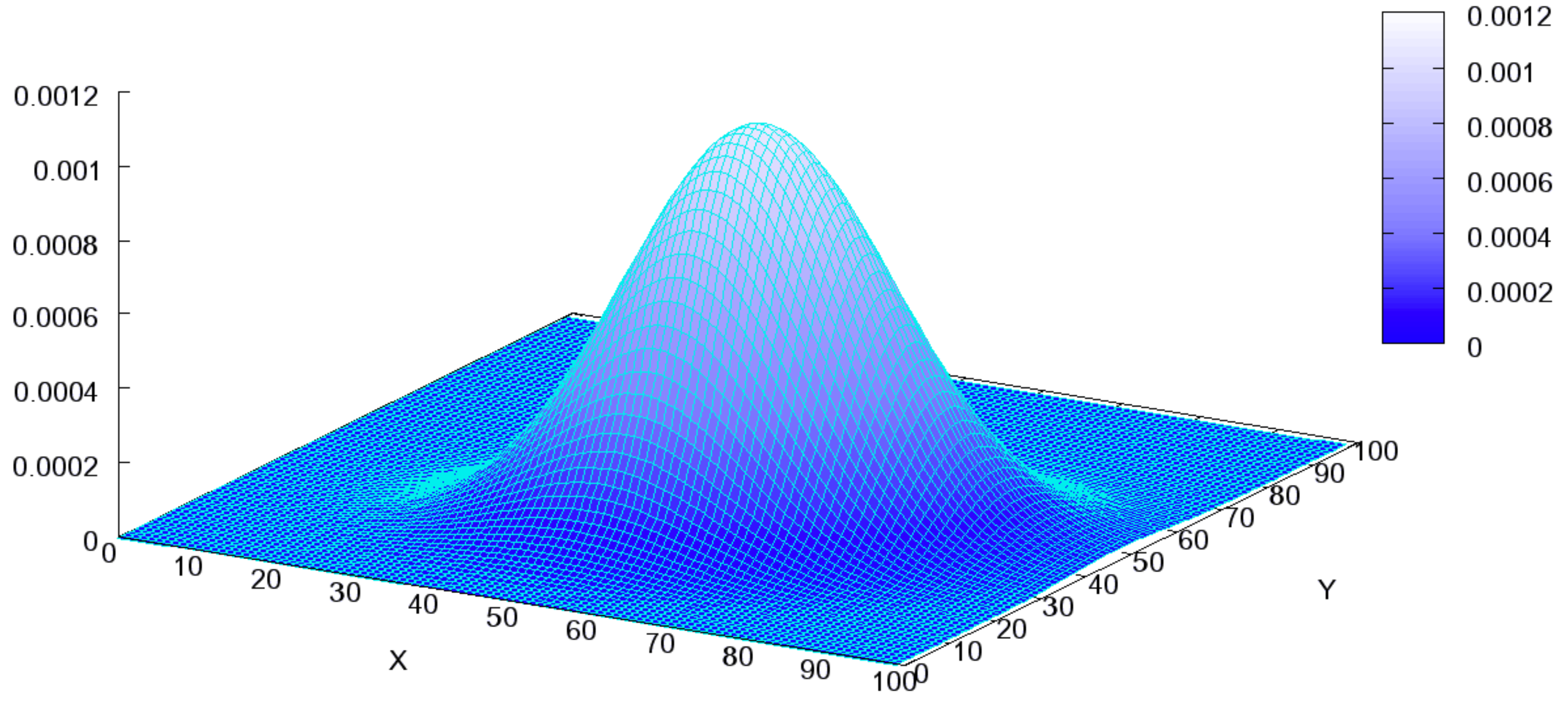


https://en.wikipedia.org/wiki/Multivariate_normal_distribution

Aside: the multivariate Gaussian distribution

“Information” form / natural parameters

$$p(x) = (2\pi)^{-D/2} \exp\left\{ -\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu) \right\}$$



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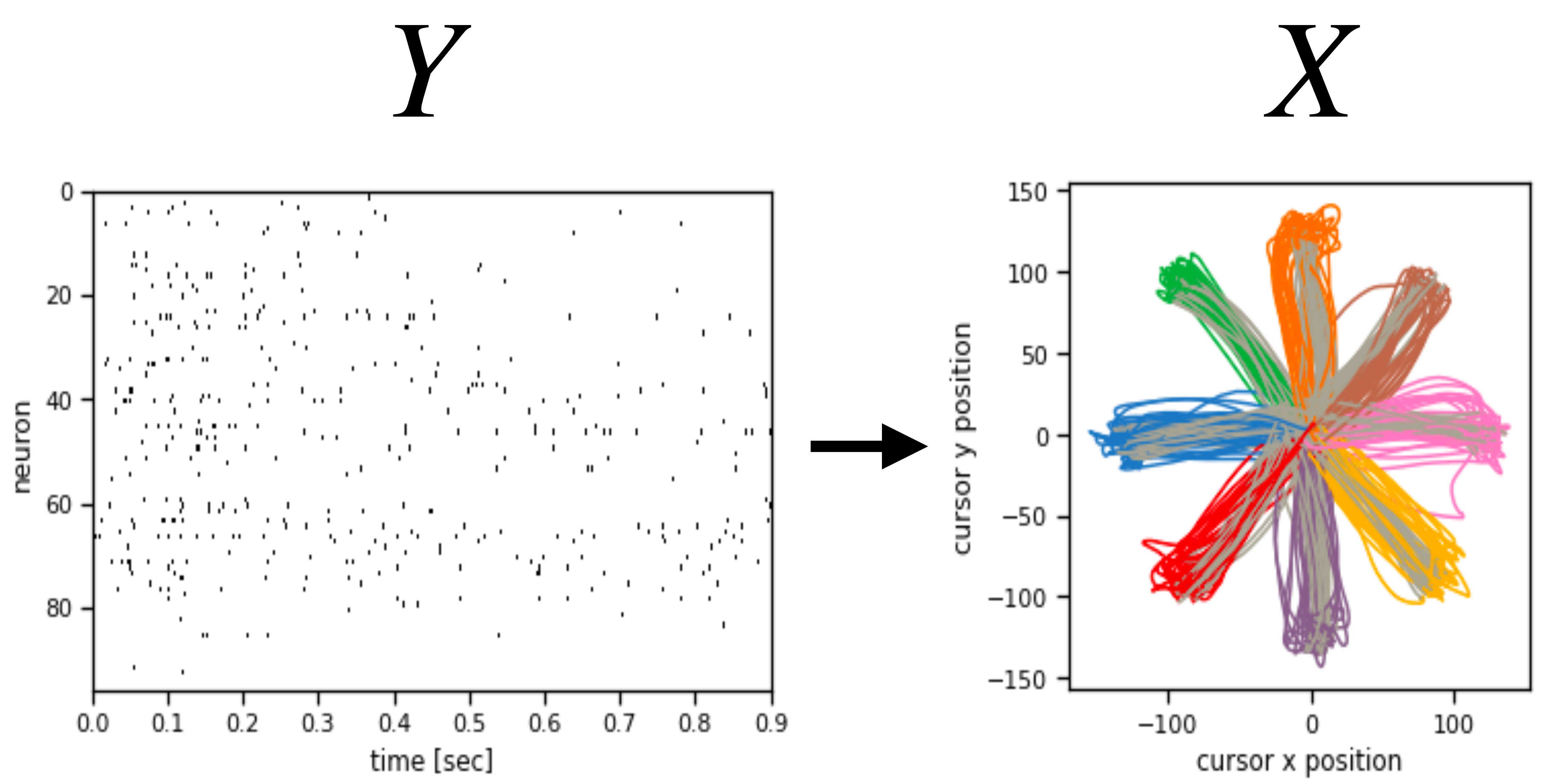
Decoding movement from neural spike trains

Deriving the posterior (decoder)

$$p(X \mid Y) \propto \prod_{t=1}^{T} \left[p(y_t \mid x_t)\, p(x_t) \right]$$

$$= \prod_{t=1}^{T} \left[\prod_{n=1}^{N} \mathcal{N}(y_{tn} \mid c_n^\top x_t + d_n, r_n^2)\, \mathcal{N}(x_t \mid 0, Q) \right]$$



Improving upon the basic model

Decoding movement from neural spike trains

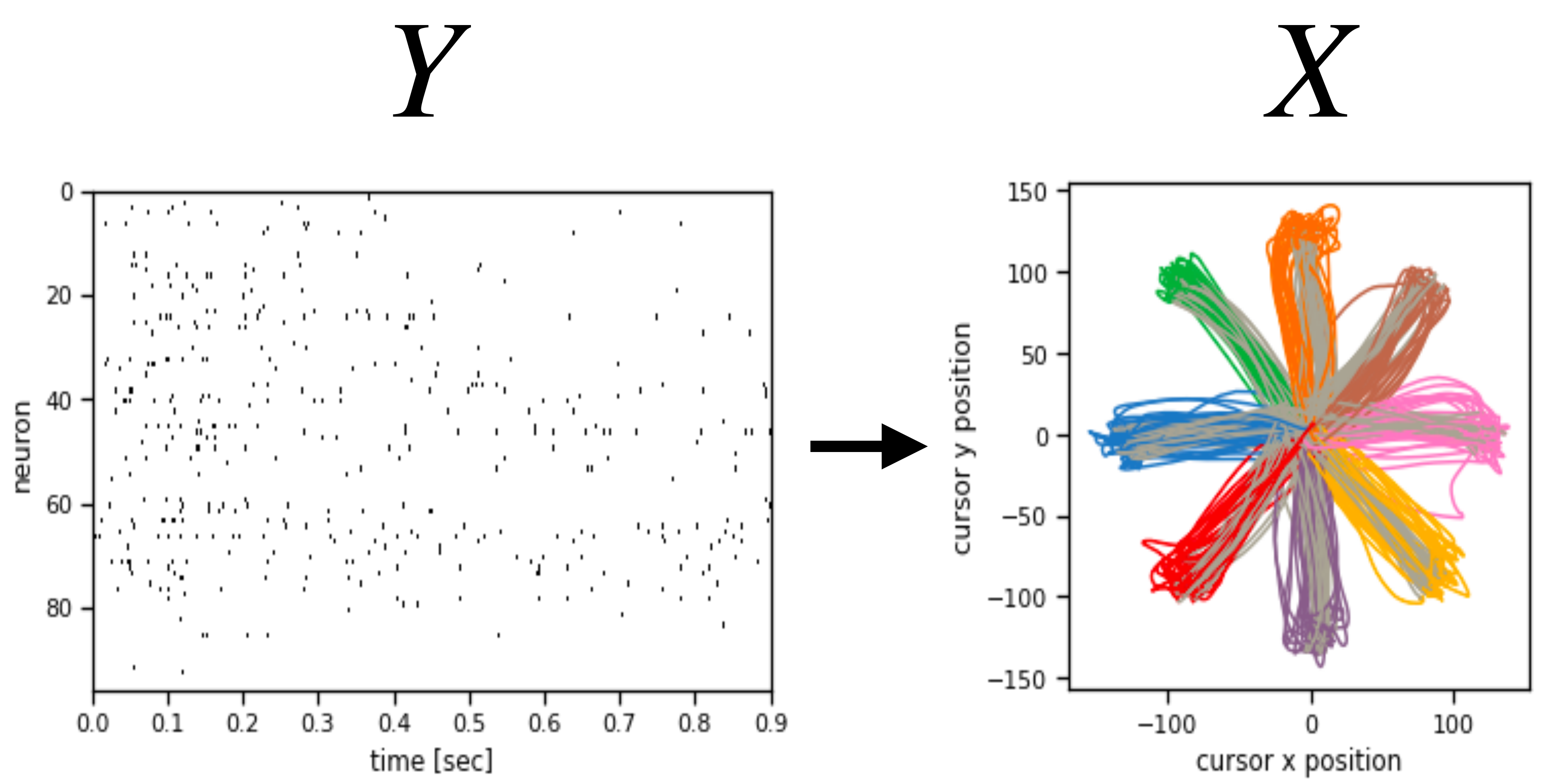
A linear dynamical system (LDS) model

- One of the problems with the basic model is that it treated each time bin as independent.
- Instead, consider the following prior

$$p(X) = p(x_1)\prod_{t=2}^{T} p(x_t \mid x_{t-1})$$

$$= \mathcal{N}(x_1 \mid 0, Q)\prod_{t=2}^{T} \mathcal{N}(x_t \mid Ax_{t-1}, Q)$$

- Parameterized by **dynamics matrix** $A \in \mathbb{R}^{D \times D}$.



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Derive the posterior under the LDS

$$p(X \mid Y) \propto \left[\mathcal{N}(x_1 \mid 0, Q)\prod_{t=2}^{T}\mathcal{N}(x_t \mid Ax_{t-1}, Q)\right]\left[\prod_{t=1}^{T}\mathcal{N}(y_t \mid Cx_t + d, R)\right]$$

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Derive the posterior under the new model (continued)

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Final results

$$p(X \mid Y) = \mathcal{N}(\mathrm{vec}(X) \mid \mu, \Sigma)$$

$$\Sigma = J^{-1} \qquad \mu = J^{-1}h$$

$$J = \begin{bmatrix} J_{11} & J_{21}^\top & & & \\ J_{21} & J_{22} & J_{32}^\top & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & J_{T,T-1}^\top \\ & & & J_{T,T-1} & J_{TT} \end{bmatrix} \qquad h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_T \end{bmatrix}$$

Where

- The diagonal blocks are $J_{tt} = Q^{-1} + A^\top Q^{-1} A$ (except for J_{11} and J_{TT}).
- The lower diagonal blocks are $J_{t,t-1} = -Q^{-1}A$
- The linear coefficients are $h_t = C^\top R^{-1}(y_t - d)$.

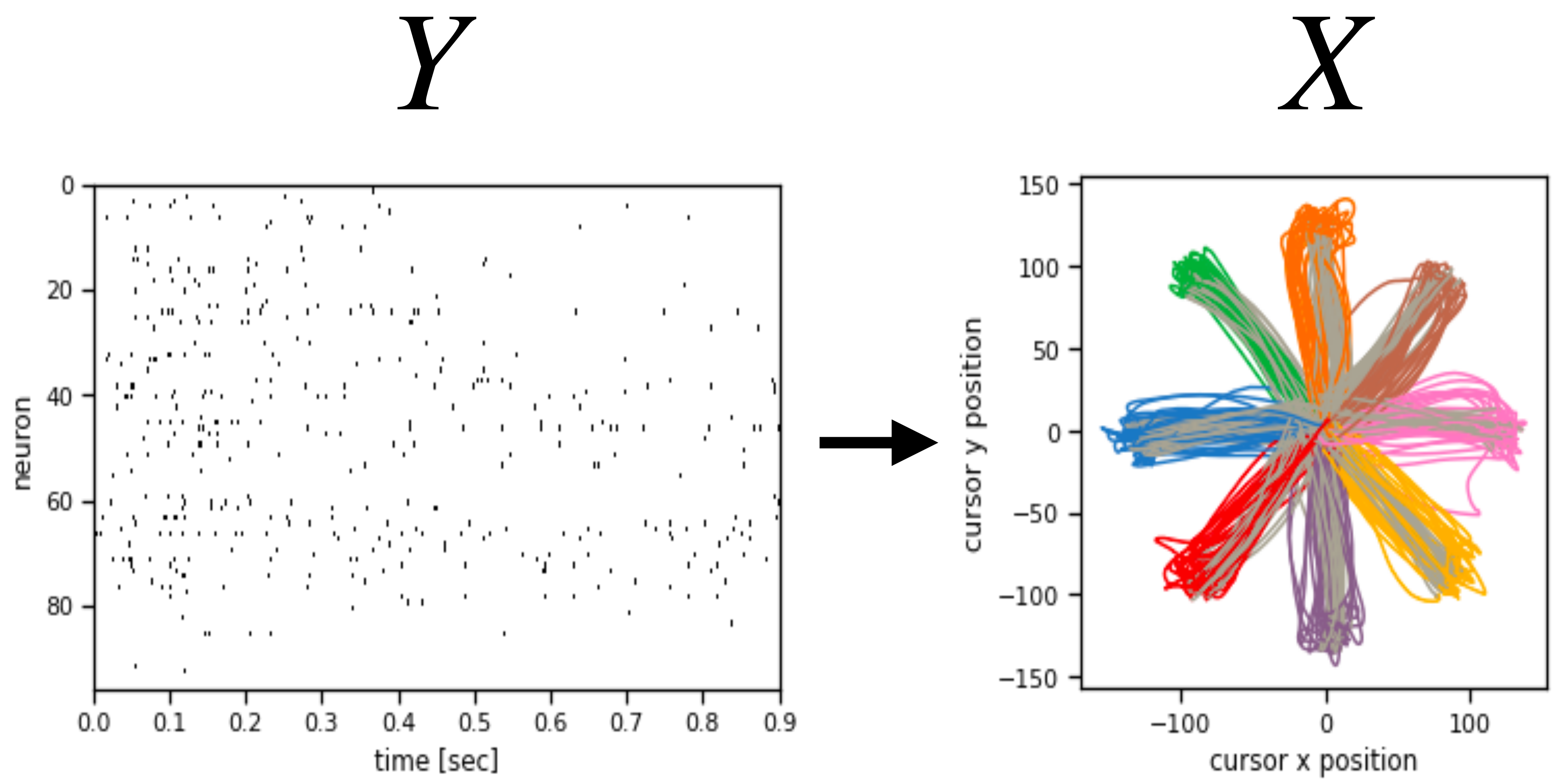
Decoding movement from neural spike trains

Poisson observations

- So far we've used a linear, Gaussian encoder for the spikes, even though they are counts!
- Suppose instead,

$$p(Y \mid X) = \prod_{t=1}^{T} \prod_{n=1}^{N} \mathrm{Po}\left(y_{tn} \mid f(c_n^\top x_t + d_n)\right)$$

- The posterior is no longer Gaussian, but it's common to approximate it as one.



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Laplace approximation

Approximate the posterior as

$$p(X \mid Y) \approx \mathcal{N}(\mu, \Sigma)$$

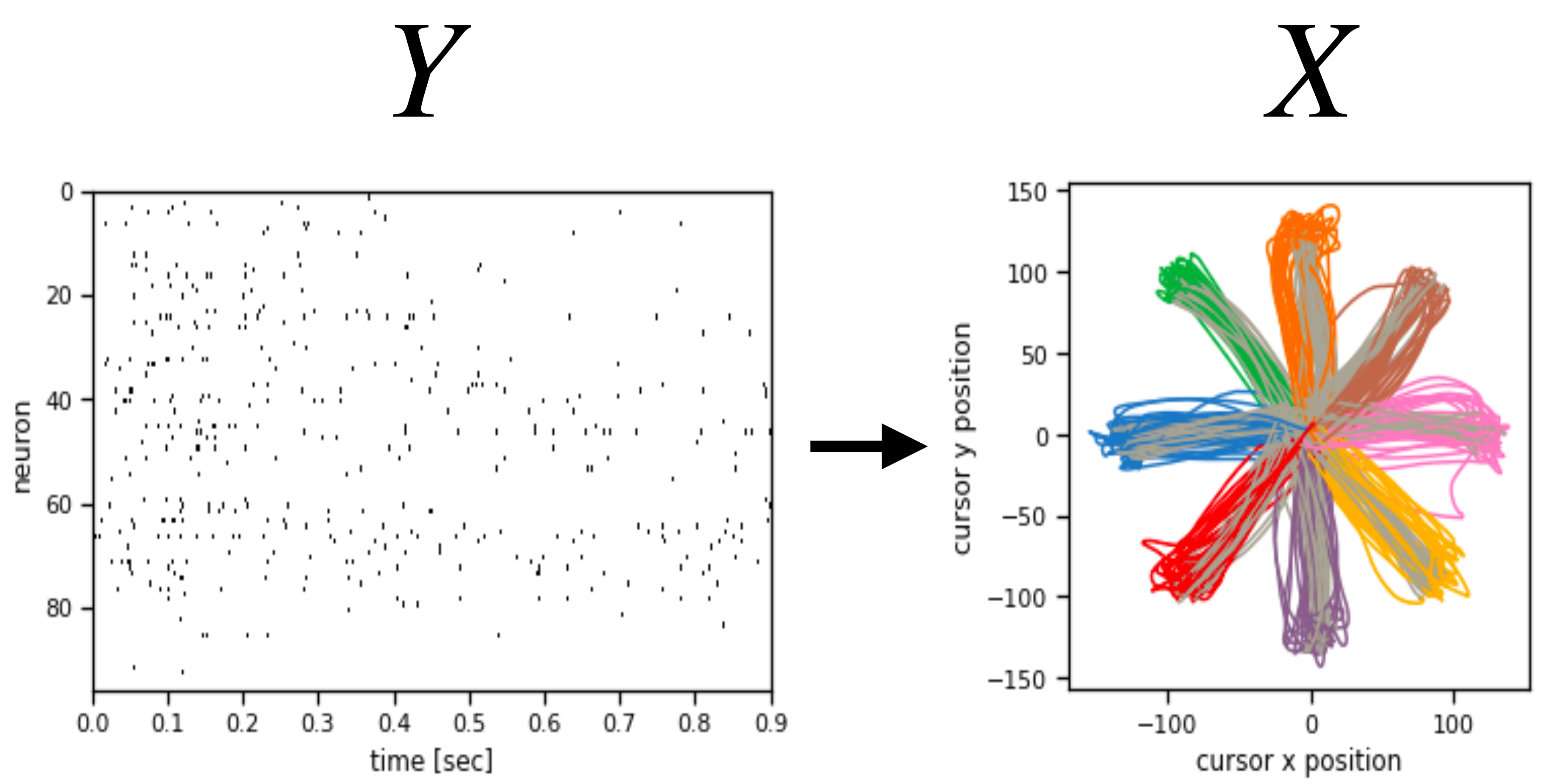
where

$$\mathcal{L}(X) = -\log p(X, Y)$$

$$\mu = \operatorname{argmin}_X \mathcal{L}(X)$$

$$\Sigma = \left[\nabla^2 \mathcal{L}(X) \Big|_{X=\mu} \right]^{-1}$$

For GLM encoders, the log joint is concave and μ and Σ can be found efficiently.



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Laplace approximation under a Poisson GLM encoder

Derive the Hessian under the Poisson GLM encoder

$$-\log p(Y \mid X) = -\sum_{t=1}^{T}\sum_{n=1}^{N} \log \mathrm{Po}\left(y_{tn} \mid f(c_n^\top x_t + d_n)\right)$$

“Direct” decoders and structured prediction

Decoding movement from neural spike trains

Structured decoders

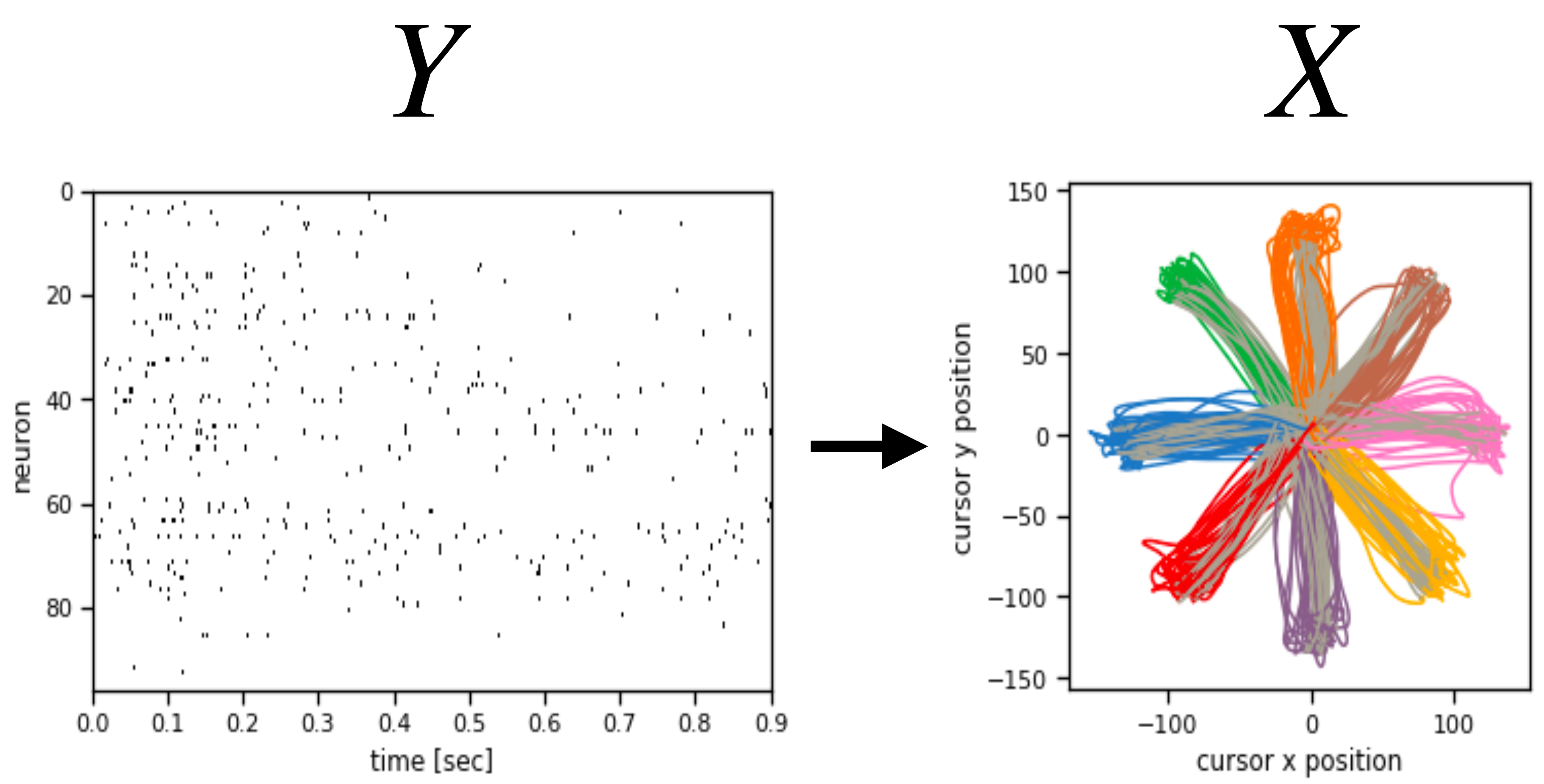
- If we're going to make a Gaussian approximation anyway, why not learn more flexible means and covariances?
- Recall the form of the LDS posterior,

$$J_{tt} = Q^{-1} + A^\top Q^{-1} A$$

$$J_{t,t-1} = -Q^{-1} A$$

$$h_t = C^\top R^{-1}(y_t - d)$$

- **Idea**: replace these with learned functions of $y_{1:T}$.



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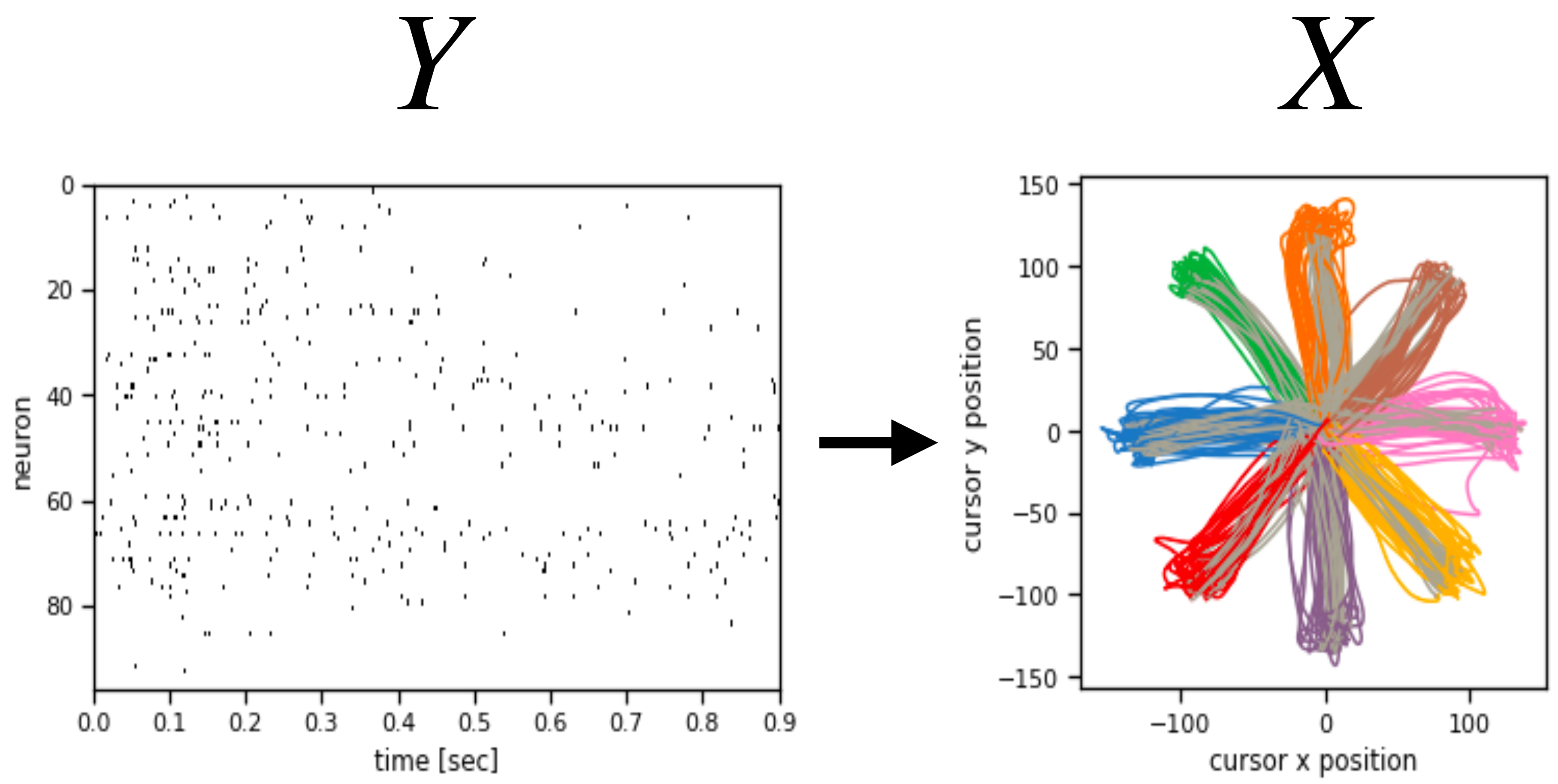
Structured decoders

- For example,

$$p(X \mid Y) = \mathcal{N}(\text{vec}(X) \mid \mu, \Sigma)$$

- $$\mu = J(Y)^{-1}h(Y)$$ $$\Sigma = J(Y)^{-1}$$

- Where $J(Y)$ is composed of blocks $J_{tt}(y_{t-\Delta:t+\Delta})$, $J_{t,t-1}(y_{t-\Delta:t+\Delta})$ and $h(Y)$ is composed of blocks $h_t(y_{t-\Delta:t+\Delta})$.



Conclusion

- Decoding and encoding are two sides of the same coin.
- We can treat decoding as a simple regression problem, but sometimes we have prior information about X or the encoder $p(Y \mid X)$ that we can leverage.
- Bayesian rule tells us how to combine prior and likelihood to derive a posterior distribution.
- However, the posterior rarely has a simple, closed form, so we need some approximations.
- Structured decoders give us a way to capture general dependency structure while allowing more flexible features of the data to be learned and incorporated.