Machine Learning Methods for Neural Data Analysis

Sequential VAEs

Scott Linderman

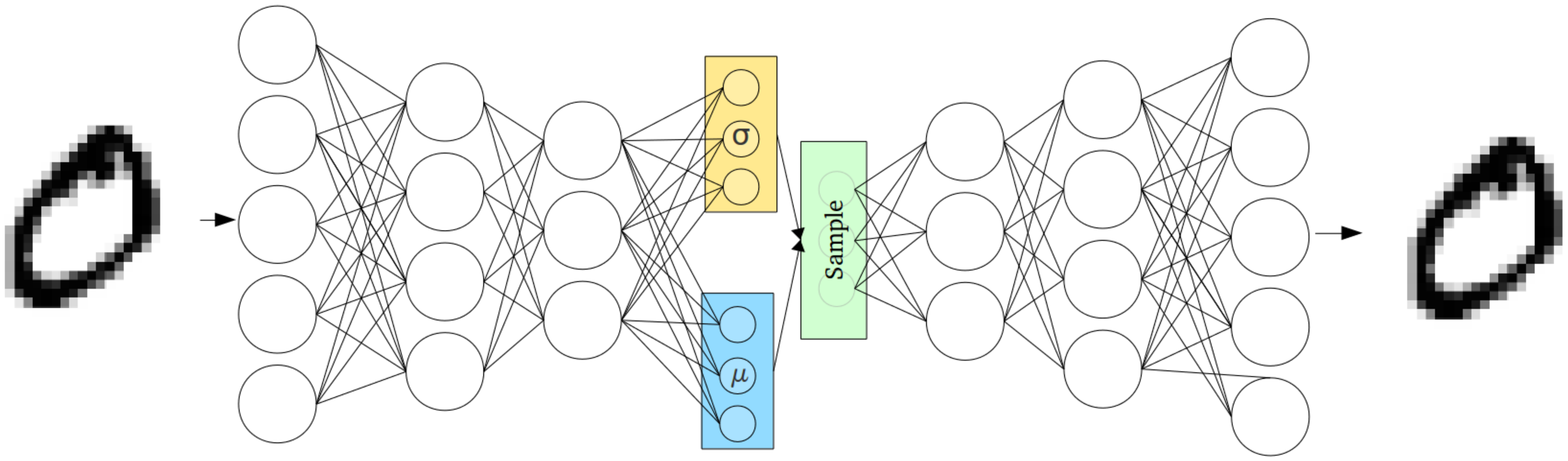
STATS 220/320 (NBIO220, CS339N).

Announcements

- In class **project presentations** next Friday (3/17).
  - ~6 minutes per presentation.
  - I will set up a dropbox/drive folder where you can upload your presentations in advance, just in case we have technical difficulties.
- Next **Monday (3/13)** we will have a **guest lecture by Prof. Russ Poldrack** (Stanford Psychology), a world expert in fMRI data analysis.
  - **There will not be a zoom link — please attend in person.**

Variational Autoencoders (VAEs)

We can generalize this approach to **nonlinear factor analysis** using neural networks; a.k.a. **variational autoencoders (VAEs)**.



Variational Autoencoders

ELBO Surgery

We can rearrange the ELBO in many ways,

$$\begin{aligned}\mathscr{L}(\theta, \phi) &= \mathbb{E}_{q(x_t)}\left[\log p(x_t, y_t; \theta) - \log q(x_t)\right] \\ &= \underbrace{\mathbb{E}_{q(x_t)}\left[\log p(y_t \mid x_t; \theta)\right]}_{\text{expected log likelihood}} - \underbrace{\mathrm{KL}\left(q(x_t) \,\|\, p(x_t; \theta)\right)}_{\text{KL to prior}}\end{aligned}$$

Applying the reparameterization trick,

$$\mathscr{L}(\theta, \phi) \approx \mathbb{E}_{\epsilon_t}\left[\log p(y_t \mid \hat{x}_t; \theta)\right] - \mathrm{KL}\left(q(x_t \mid y_t; \phi) \,\|\, p(x_t; \theta)\right)$$

Variational Autoencoders

ELBO Surgery

Under a Gaussian model

$$\begin{aligned}\mathscr{L}(\theta, \phi) &= \mathbb{E}_{\epsilon_t}\left[\log p(y_t \mid \hat{x}_t; \theta)\right] - \mathrm{KL}\left(q(x_t \mid y_t; \phi) \,\|\, p(x_t; \theta)\right) \\ &= \underbrace{-\frac{1}{2\sigma^2}\|y_t - \hat{y}_t\|_2^2}_{\text{reconstruction loss}} - \mathrm{KL}\left(q(x_t \mid y_t; \phi) \,\|\, p(x_t; \theta)\right) + c\end{aligned}$$

Variational Autoencoders

Amortization and Approximation gaps

- When we switch to nonlinear models, the posterior is no longer Gaussian ⇒ **approximation gap**
- Moreover, neural network encoder may not produce the best Gaussian approximation ⇒ **amortization gap**.
- Both lead to suboptimal inference and learning.

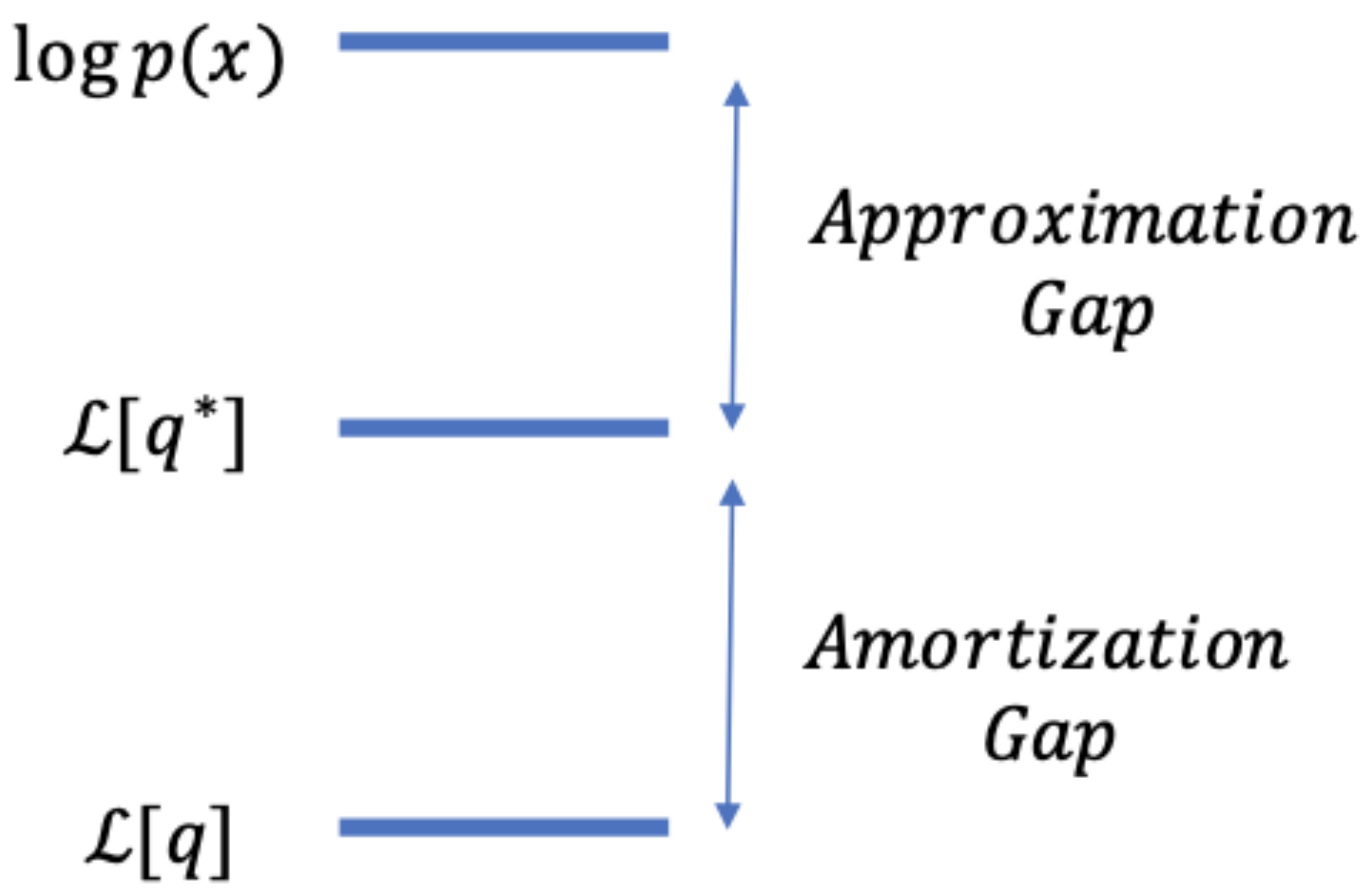


Figure 1. Gaps in Inference

Sequential VAEs

VAEs for time series data

- In neuroscience, we're often interested in **sequential data** $y_{1:T} = (y_1, \ldots, y_T)$.
  - For example, neural spike trains or behavioral time series.
- We could model each time point an an independent observation,

$$x_t \sim \mathcal{N}(0, I) \quad y_t \sim \mathcal{N}(f(x_t; \theta), \sigma^2 I)$$

where $f(x; \theta)$ is a neural network with weights θ, as in a VAE.

- Can we do better?

Sequential VAEs

- We could incorporate **temporal dependencies into the prior.** E.g., via an linear dynamical system prior,

$$p(x_{1:T}) = \mathcal{N}(x_1 \mid 0, Q_1) \prod_{t=2}^{T} \mathcal{N}(x_t \mid A x_{t-1} + b, Q).$$

- More generally, we could have a **nonlinear dynamical system**,

$$p(x_{1:T}) = \mathcal{N}(x_1 \mid 0, Q_1) \prod_{t=2}^{T} \mathcal{N}(x_t \mid h(x_{t-1}; \theta), Q).$$

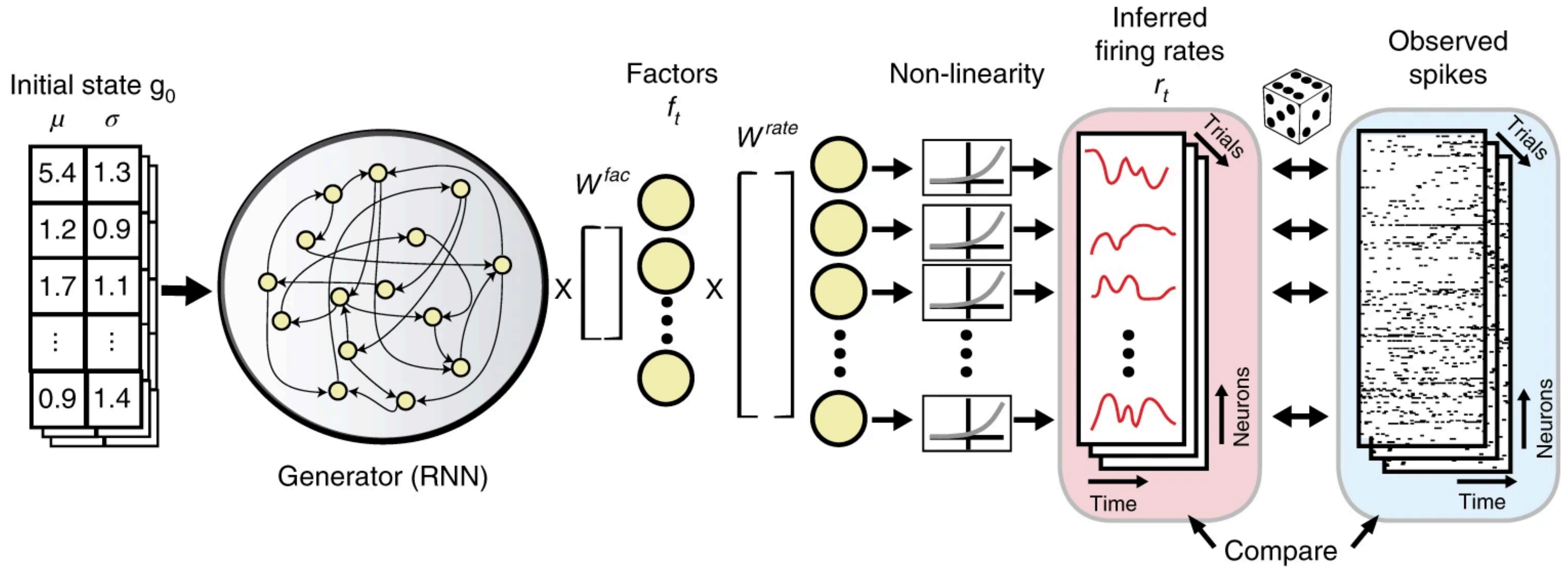
  where θ are the parameters of a neural network.

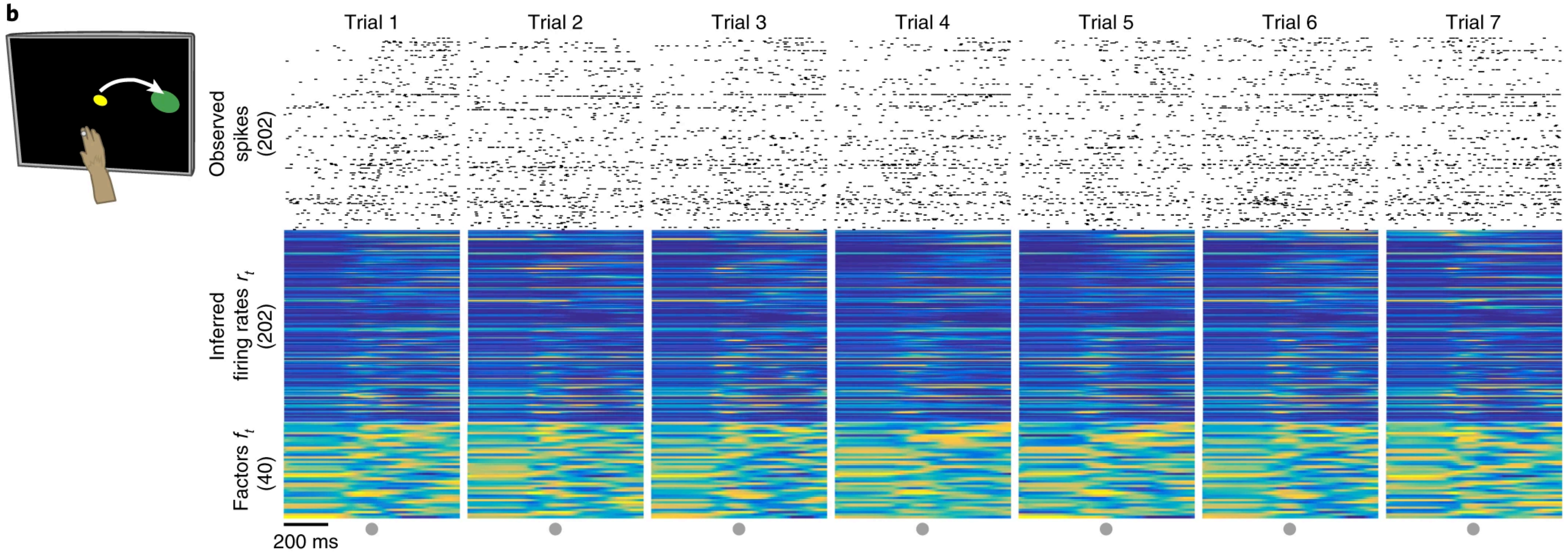
- For example, $h(x; \theta)$ could be a **recurrent neural network.**

Stochastic RNNs

LFADS: Latent Factor Analysis for Dynamical Systems

- LFADS uses a recurrent neural network (the **generator**) to model nonlinear dynamics of neural activity.
- In the basic model, the RNN has **deterministic dynamics** with a **random initial condition.**
- The RNN state is mapped through a **GLM** to obtain firing rates for a **Poisson model.**



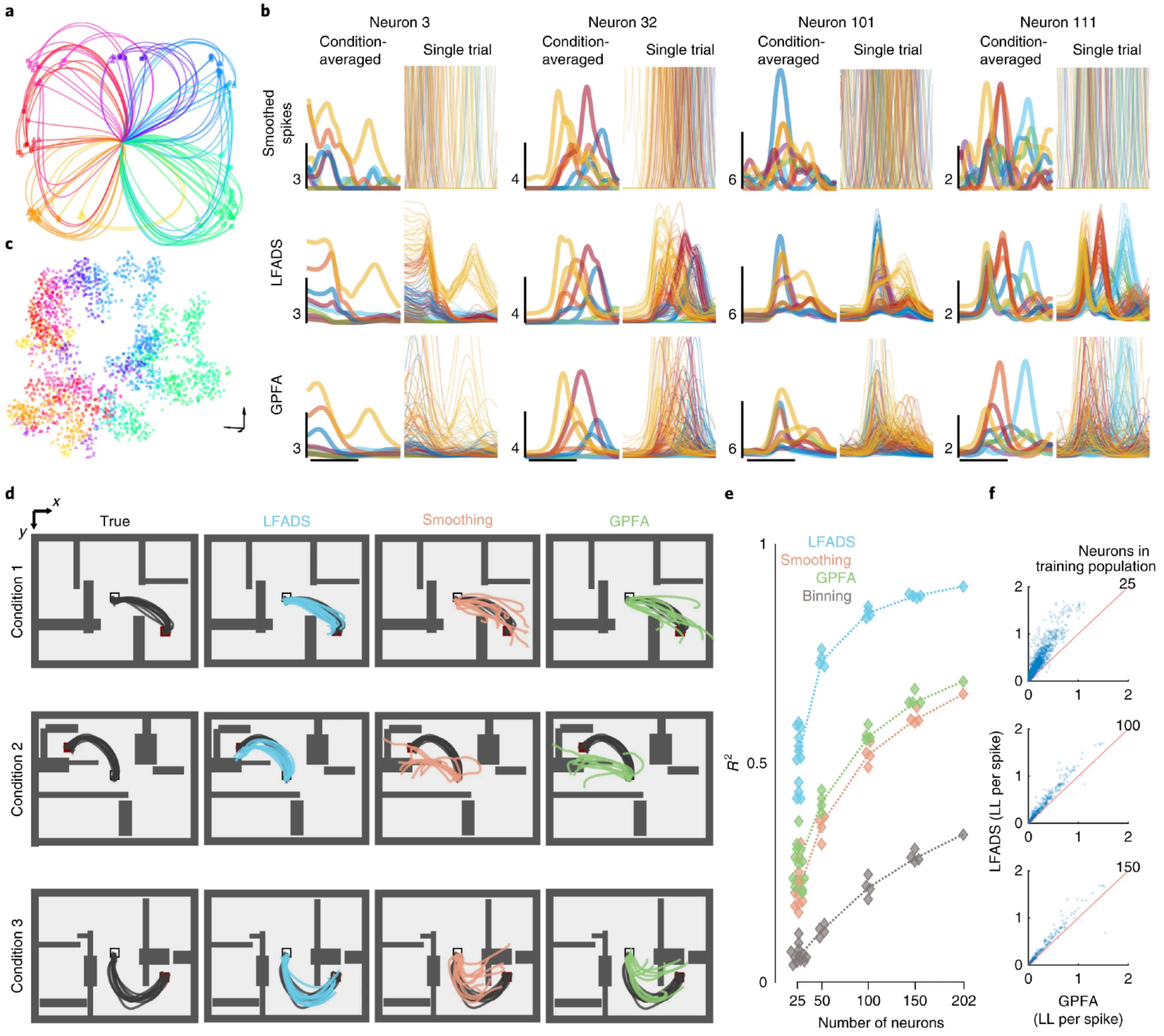


Pandarinath et al (2018)

Stochastic RNNs

LFADS: Latent Factor Analysis for Dynamical Systems

- LFADS learns accurate **single-trial firing rates** and achieves **state-of-the-art decoding performance** on monkey reaching tasks (Recall Lab 6).



Pandarinath et al (2018)

Sequential VAEs

Stochastic dynamics vs stochastic inputs

- LFADS uses a slightly different formulation of the prior.
- Instead of having **stochastic dynamics**,

$$p(x_{1:T}) = \mathcal{N}(x_1 \mid 0, Q_1) \prod_{t=2}^{T} \mathcal{N}(x_t \mid h(x_{t-1}; \theta), Q).$$

  It uses **stochastic inputs** with **deterministic dynamics.**

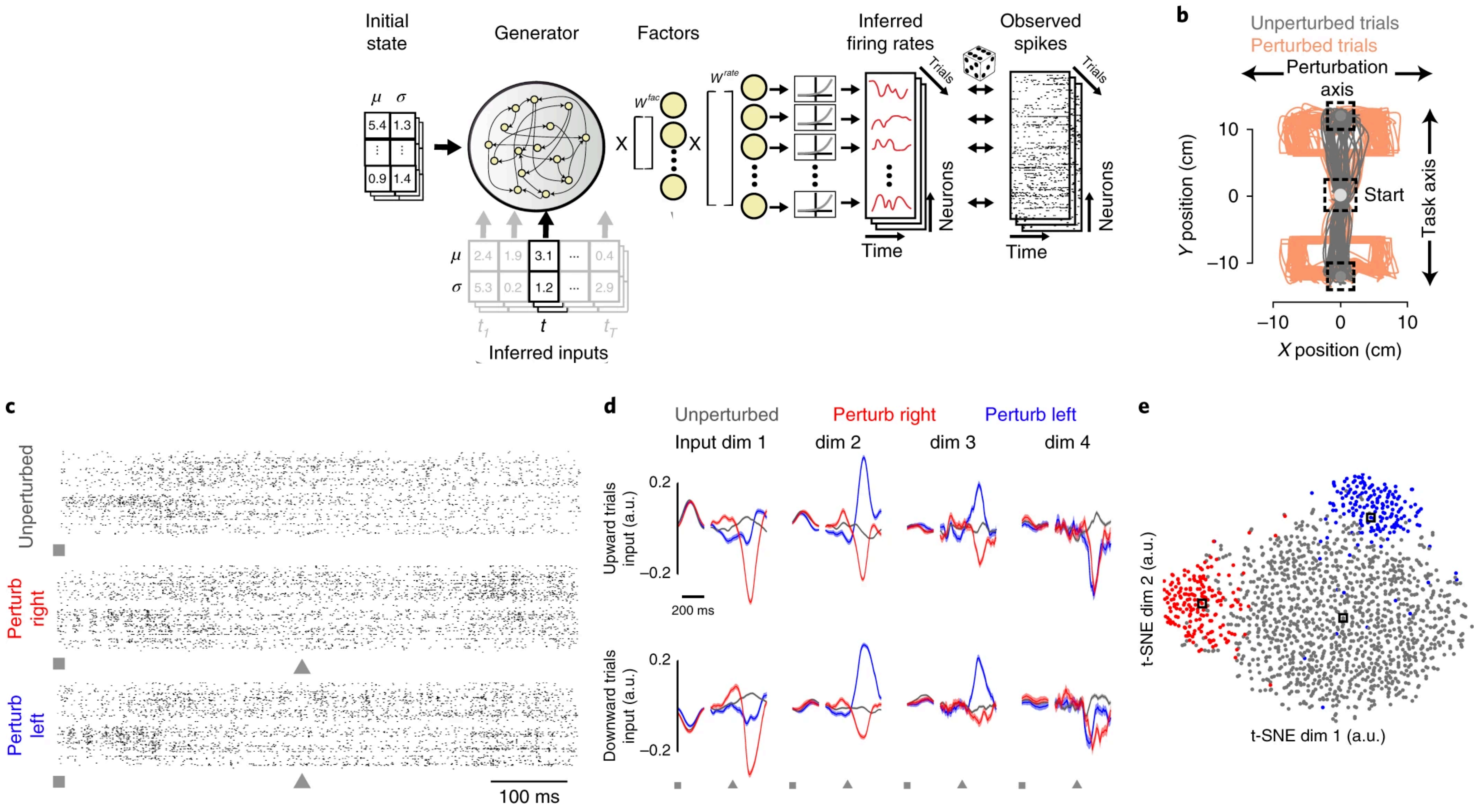
$$x_0 \sim \mathcal{N}(\mid 0, Q_1) \qquad u_t \overset{\text{iid}}{\sim} \mathcal{N}(0, I) \qquad x_t = h(x_{t-1}, u_t; \theta).$$

- This is just a **reparameterization**. It implies a distribution on $x_{0:T}$, but that distribution could be quite complex since h is nonlinear.

Stochastic RNNs

LFADS: Latent Factor Analysis for Dynamical Systems

- The **inferred inputs** can suggest the presence, identity, and timing of **unexpected changes** in the dynamics.
- For example, in trials where the **cursor was randomly perturbed** to the right or left, inputs capture corresponding changes in neural activity.



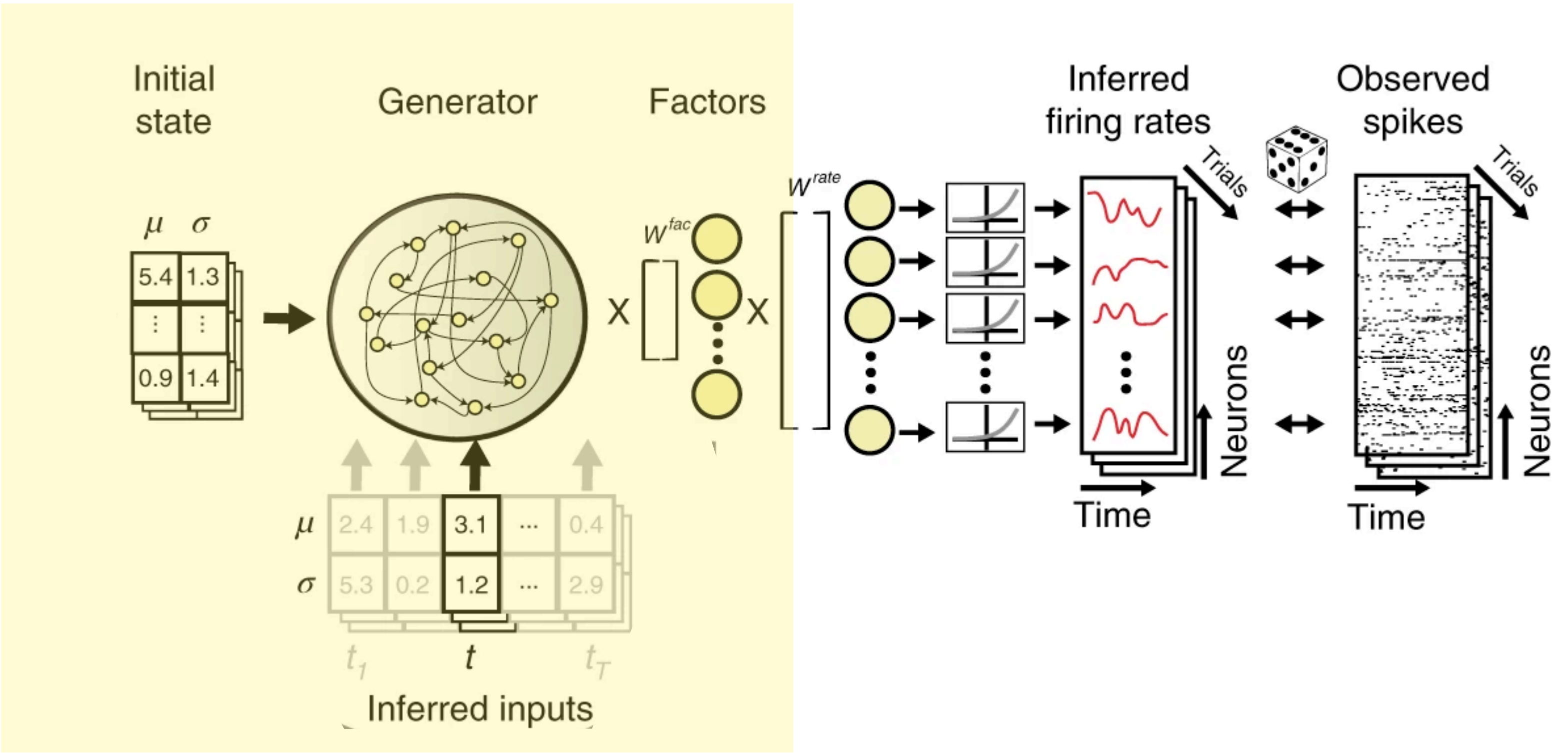
Pandarinath et al (2018)

Stochastic RNNs

The LFADS probabilistic model

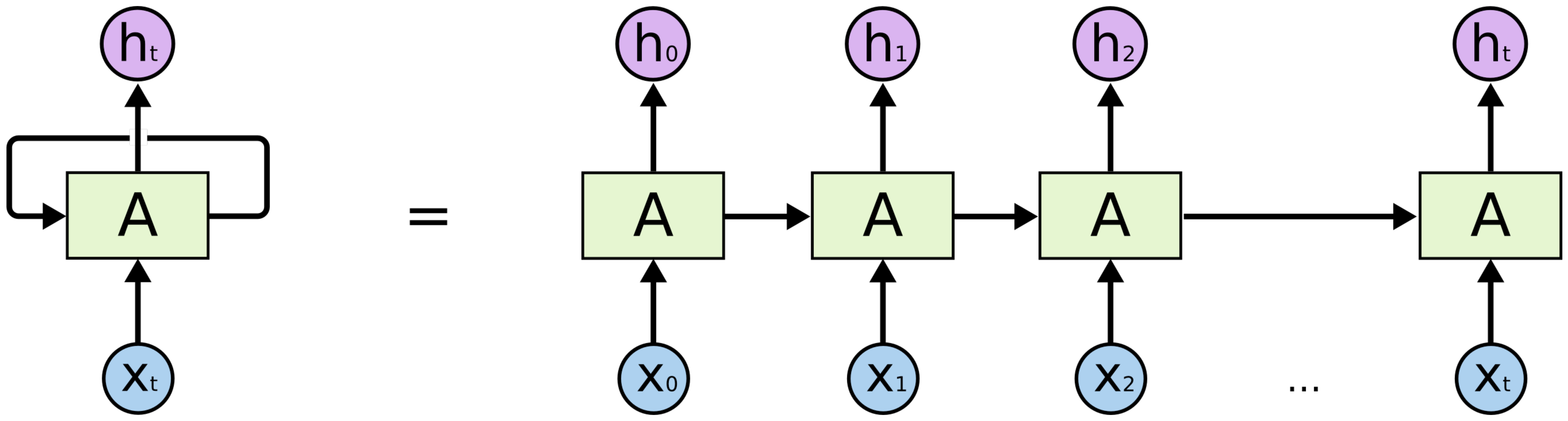
- We can unwind the recursion to write the state at time t as a deterministic function of the initial condition and the inputs up to time t,

$$
\begin{aligned}
x_t &= h(x_{t-1}, u_t, \theta) \\
&= h(h(x_{t-2}, u_{t-1}, \theta), u_t, \theta) \\
&= h(\cdots h(h(x_0, u_1, \theta), u_2, \theta) \cdots) \\
&\triangleq h_t(x_0, u_{1:t}, \theta)
\end{aligned}
$$



Sequential VAEs

“Vanilla” RNNs



https://colah.github.io/posts/2015-08-Understanding-LSTMs/

Stochastic RNNs

The LFADS probabilistic model

- To optimize the ELBO, we'll need derivatives of the state with respect to the inputs,

$$\frac{\partial x_t}{\partial x_0} = \frac{\partial}{\partial x_{t-1}} h(x_{t-1}, u_t, \theta) \cdot \frac{\partial x_{t-1}}{\partial x_0}$$

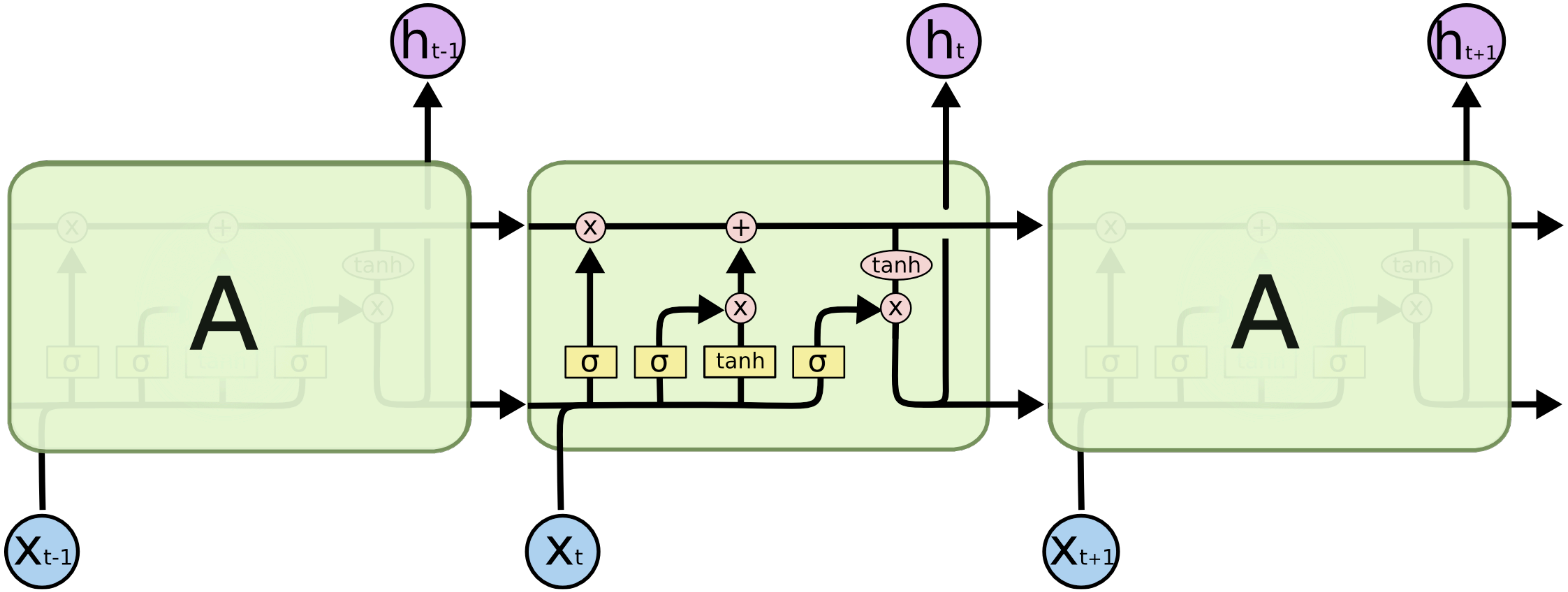
- In a vanilla RNN, $h(x, u) = g(Wx + Bu)$ where $g(\cdot)$ is an element-wise nonlinearity like $\tanh$ or relu. Then,

$$\frac{\partial}{\partial x_{t-1}} h(x_{t-1}, u_t, \theta) = \mathrm{diag}(g'(Wx_{t-1} + Bu_t))\, W$$

- Multiplying a bunch of these matrices together leads to **vanishing gradients.**

Sequential VAEs

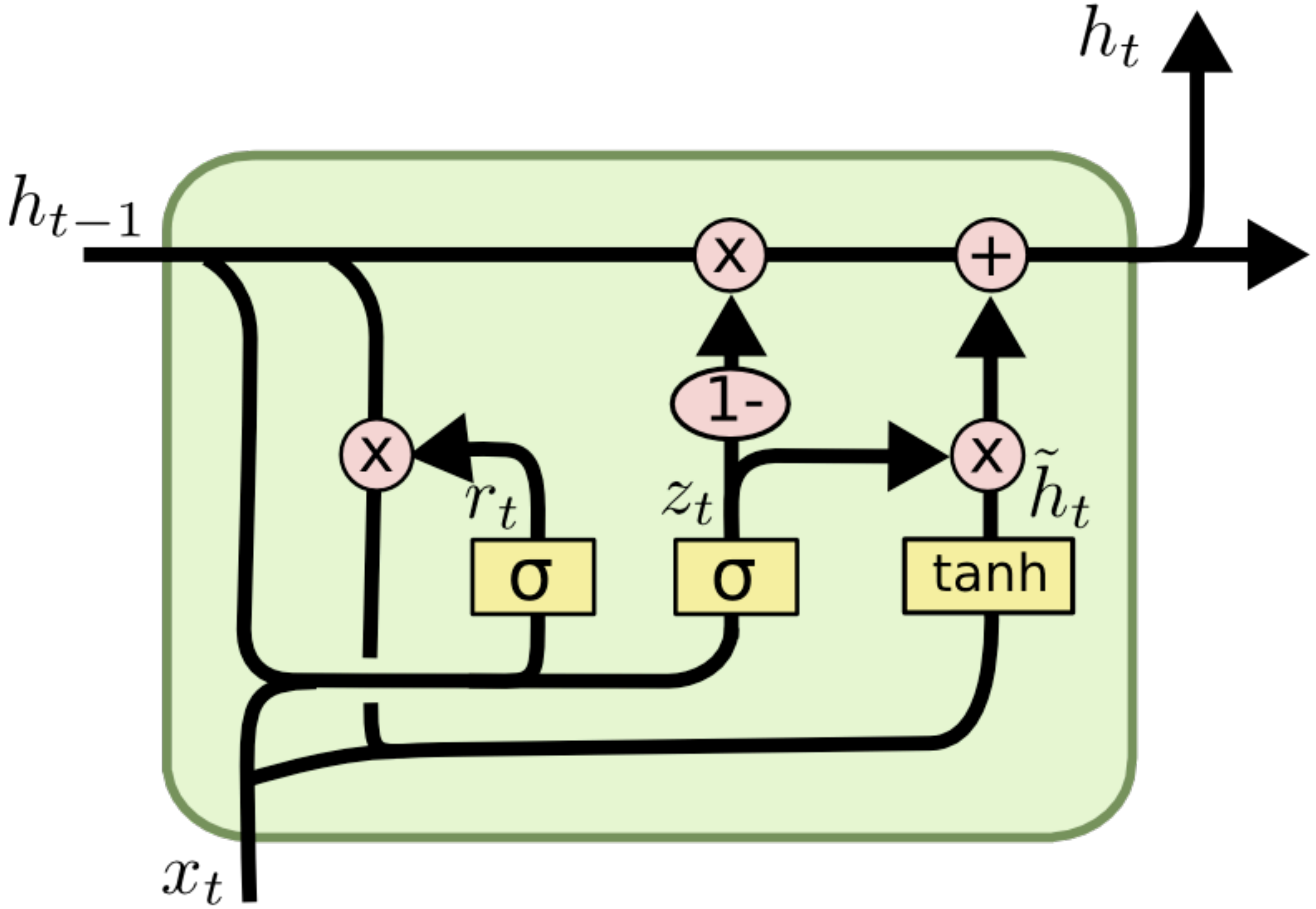
Long Short-Term Memory (LSTM) networks



https://colah.github.io/posts/2015-08-Understanding-LSTMs/

Sequential VAEs

Gated Recurrent Units (GRUs)



$$z_t = \sigma\left(W_z \cdot [h_{t-1}, x_t]\right)$$

$$r_t = \sigma\left(W_r \cdot [h_{t-1}, x_t]\right)$$

$$\tilde{h}_t = \tanh\left(W \cdot [r_t * h_{t-1}, x_t]\right)$$

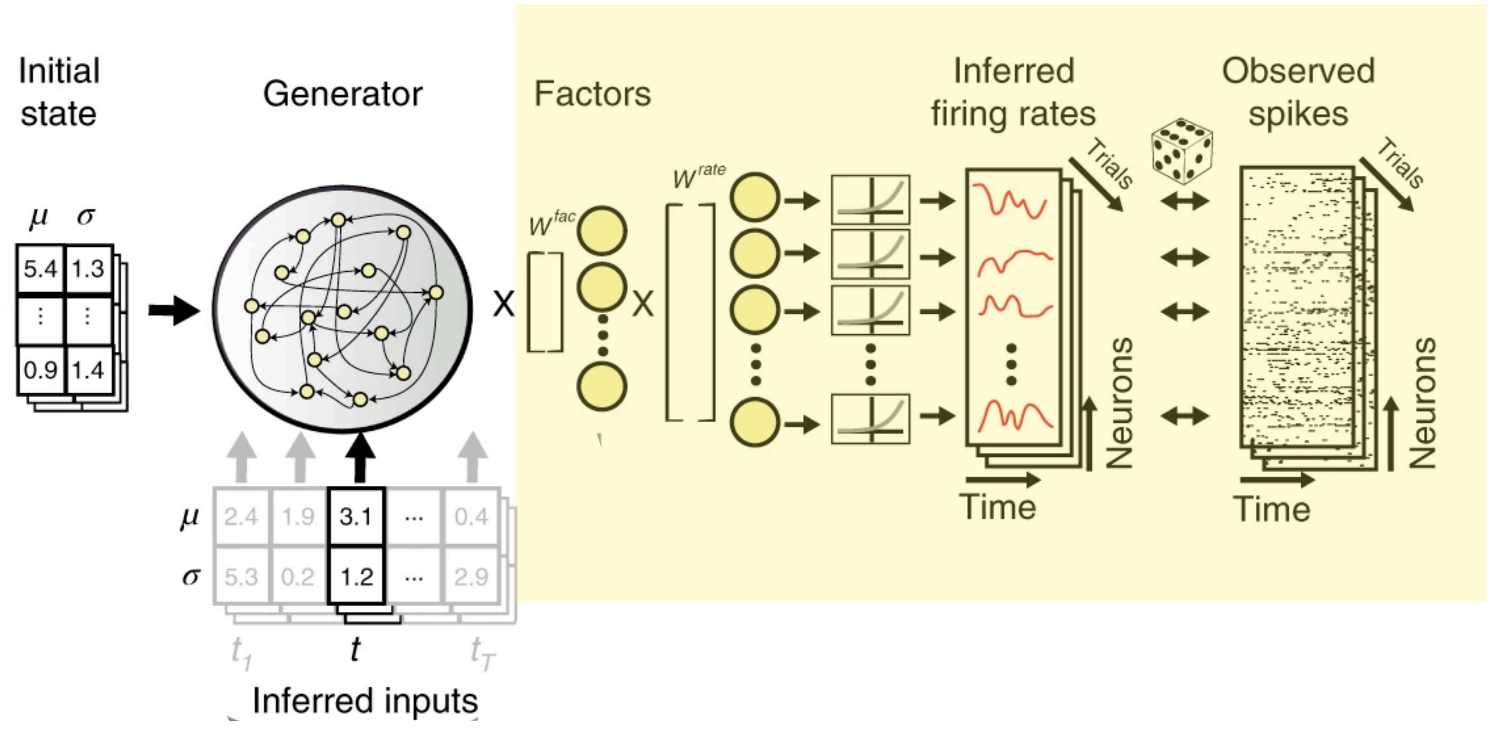
$$h_t = (1 - z_t) * h_{t-1} + z_t * \tilde{h}_t$$

https://colah.github.io/posts/2015-08-Understanding-LSTMs/

Stochastic RNNs

The LFADS probabilistic model

- The output is modeled as a (typically simple) function of the latent state,



$$y_t \sim \mathrm{Po}\big(f(x_t)\big)$$

where, e.g.,

$$f(x_t) = \exp\left\{Cx_t + d\right\}.$$

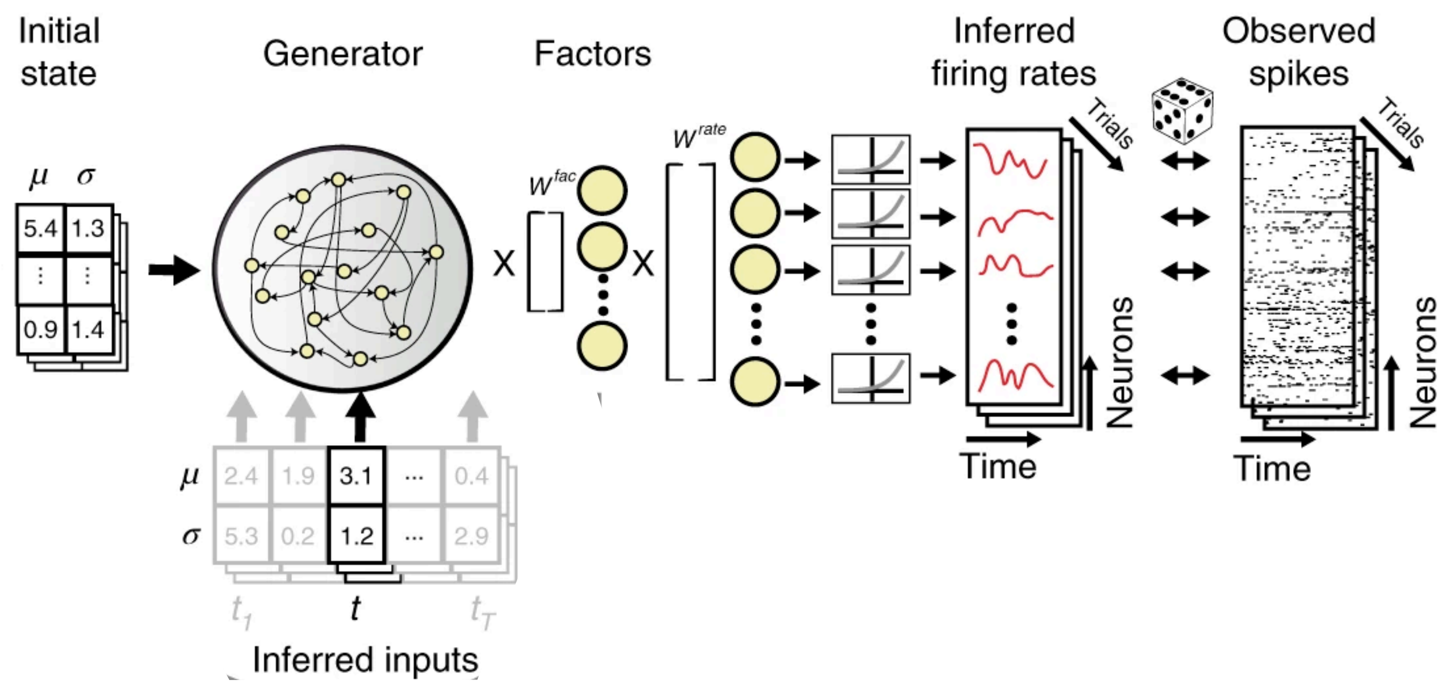
Stochastic RNNs

The LFADS probabilistic model

- Assume the initial condition and inputs have standard normal priors.

- The joint distribution is,

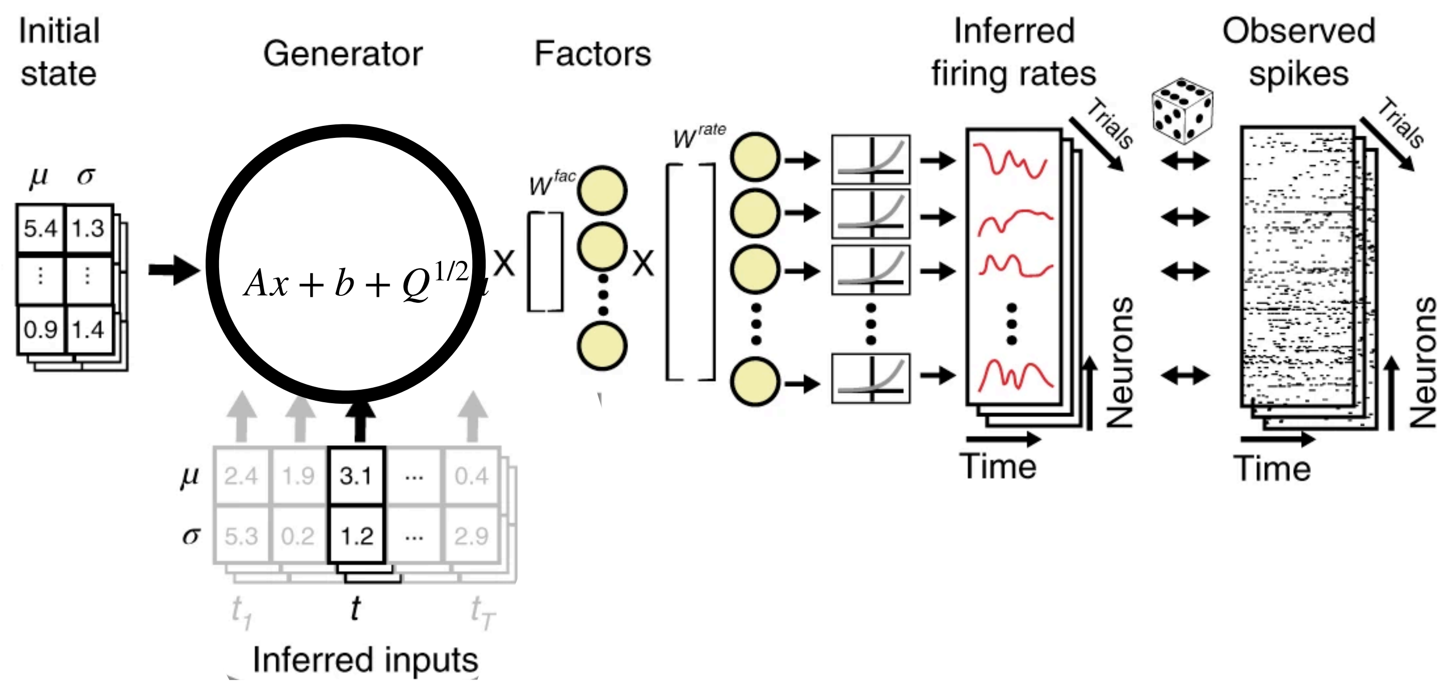
$$
\begin{aligned}
p(x_0, u_{1:T}, y_{1:T} \mid \theta) &= \mathcal{N}(x_0 \mid 0, I) \prod_{t=1}^{T} \mathcal{N}(u_t \mid 0, I)\, \mathrm{Po}(y_t \mid f(x_t)) \\
&= \mathcal{N}(x_0 \mid 0, I) \prod_{t=1}^{T} \mathcal{N}(u_t \mid 0, I)\, \mathrm{Po}\big(y_t \mid f(h_t(x_0, u_{1:t}, \theta))\big)
\end{aligned}
$$



Stochastic RNNs

Poisson LDS as a special case of LFADS

- We can view the **Poisson LDS** (c.f. Macke et al, 2011) as a special case of LFADS with a **linear generator**.



$$x_t \sim \mathcal{N}(Ax_{t-1} + b, Q) \qquad\qquad x_t = h(x_{t-1}, u_t, \theta)$$

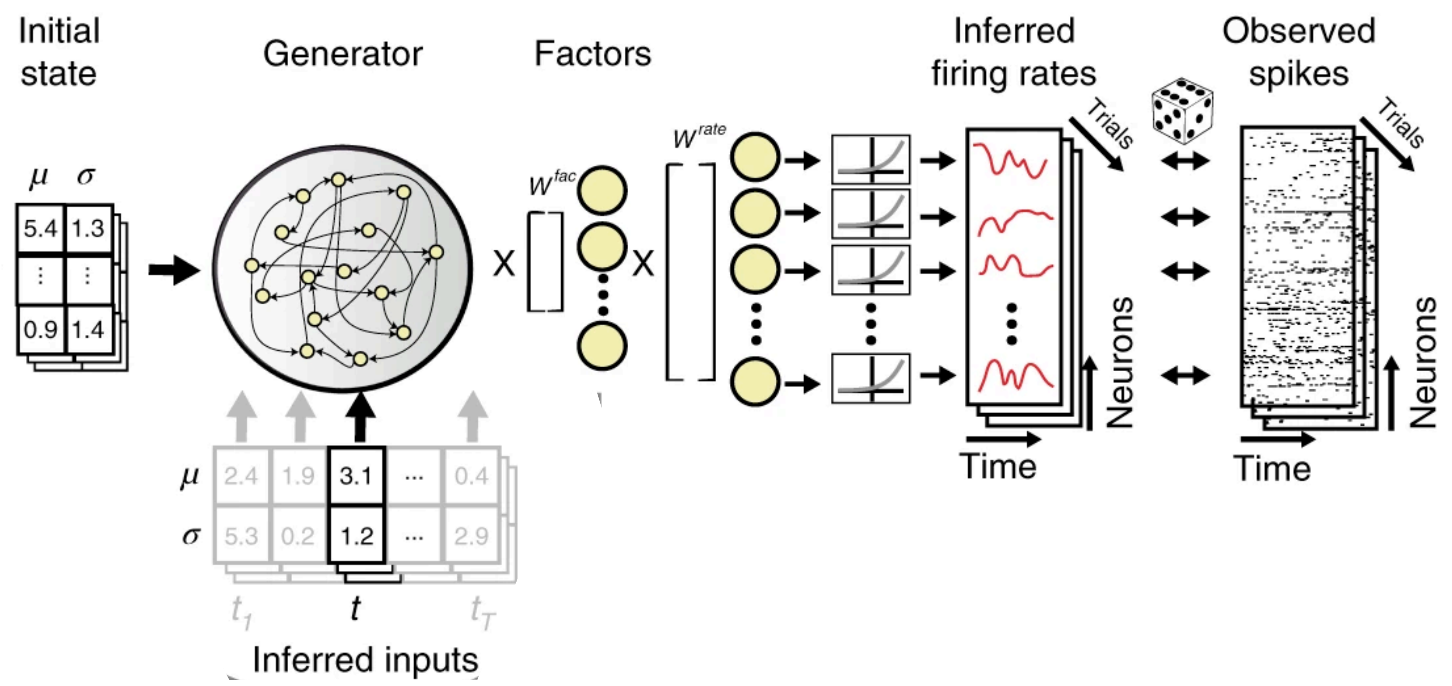
$$h(x_{t-1}, u_t, \theta) = Ax_{t-1} + b + Q^{1/2}u_t$$

$$\iff \qquad u_t \sim \mathcal{N}(0, I)$$

$$y_t \sim \mathrm{Po}\big(f(x_t)\big) \qquad\qquad y_t \sim \mathrm{Po}\big(f(x_t)\big)$$

Stochastic RNNs

LFADS learning and inference



- How to learn the parameters θ and infer the latent variables $x_0, u_{1:T}$?
- **Variational EM**:
  - **E step:** Approximate the posterior with,

$$q(x_0, u_{1:T}) \approx p(x_0, u_{1:T} \mid y_{1:T}, \theta)$$

  - **M step:** Find parameters that maximize the ELBO

$$\mathscr{L}[q, \theta] = \mathbb{E}_{q(x_0, u_{1:T})}\left[\log p(x_0, u_{1:T}, y_{1:T}) - \log q(x_0, u_{1:T})\right]$$

Stochastic RNNs

LFADS learning and inference

- Let's assume a Gaussian form for each factor,

$$q(x_0, u_{1:T}; \lambda) = \mathcal{N}(x_0 \mid \tilde{\mu}_0, \tilde{\Sigma}_0) \prod_{t=1}^{T} \mathcal{N}(u_t \mid \tilde{\mu}_t, \tilde{\Sigma}_t)$$

- This approximation is parameterized by **variational parameters** $\lambda \triangleq \{\tilde{\mu}_t, \tilde{\Sigma}_t\}_{t=0}^{T}$.

- Let $\mathscr{L}(\lambda, \theta) = \mathscr{L}[q(x_0, u_{1:T}; \lambda), \theta]$ denote the ELBO as a function of the variational and generative model parameters.

Stochastic RNNs

LFADS learning and inference

ELBO Surgery*: we can rewrite the ELBO as,

$$
\begin{aligned}
\mathscr{L}(\lambda, \Theta) &= \mathbb{E}_{q(x_0, u_{1:T}, \lambda)} \left[\log p(x_0, u_{1:T}) + \log p(y_{1:T} \mid x_0, u_{1:T}, \Theta) - \log q(x_0, u_{1:T}; \lambda)\right] \\
&= \mathbb{E}_{q(x_0, u_{1:T}, \lambda)} \left[\log p(y_{1:T} \mid x_0, u_{1:T}, \Theta) - \log \frac{q(x_0; \lambda)}{p(x_0)} - \sum_{t=1}^{T} \log \frac{q(u_t; \lambda)}{p(u_t)}\right] \\
&= \underbrace{\mathbb{E}_{q(x_0, u_{1:T}, \lambda)} \left[\sum_{t=1}^{T} \log p(y_t \mid x_0, u_{1:t}, \Theta)\right]}_{\text{expected log likelihood}} \underbrace{- \mathrm{KL}(q(x_0; \lambda) \,\|\, p(x_0)) - \sum_{t=1}^{T} \mathrm{KL}(q(u_t; \lambda) \,\|\, p(u_t))}_{\text{KL to the prior}}
\end{aligned}
$$

*For more ways of rewriting the ELBO, see Johnson and Hoffman (2017)

Stochastic RNNs

LFADS learning and inference: gradients wrt θ

Gradient ascent on the ELBO:

$$\nabla_\theta \mathscr{L}(\lambda, \theta) = \mathbb{E}_{q(x_0, u_{1:T}, \lambda)} \left[\sum_{t=1}^{T} \nabla_\theta \log p(y_t \mid x_0, u_{1:t}, \theta) \right]$$

Since the generative parameters don't appear in q, we can **pull the gradient inside the expectation** and compute it with **automatic differentiation** for any $x_0, u_{1:t}, \theta$.

Then approximate the expectation with **Monte Carlo:**

$$\nabla_\Theta \mathscr{L}(\lambda, \theta) \approx \frac{1}{M} \sum_{m=1}^{M} \left[\sum_{t=1}^{T} \nabla_\Theta \log p(y_t \mid x_0^{(m)}, u_{1:t}^{(m)}, \theta) \right] \qquad x_0^{(m)} \sim q(x_0; \lambda),\ u_t^{(m)} \sim q(u_t; \lambda).$$

Stochastic RNNs

LFADS learning and inference: the “reparameterization trick”

The gradients with respect to the variational parameters are a bit trickier:

$$\nabla_\lambda \mathscr{L}(\lambda, \theta) = \nabla_\lambda \mathbb{E}_{q(x_0, u_{1:T}, \lambda)} \left[\sum_{t=1}^{T} \log p(y_t \mid x_0, u_{1:t}, \theta) \right] - \nabla_\lambda \mathrm{KL}\big(q(x_0, u_{1:T}, \lambda) \,\|\, p(x_0, u_{1:T})\big)$$

Note that $x_0 \sim \mathscr{N}(\tilde{\mu}_0, \tilde{\Sigma}_0) \iff x_0 = \tilde{\mu}_0 + \tilde{\Sigma}_0^{1/2} \epsilon_0$ where $\epsilon_0 \sim \mathscr{N}(0, I)$.

Stochastic RNNs

LFADS learning and inference: the "reparameterization trick"

The gradients with respect to the variational parameters are a bit trickier:

$$\nabla_\lambda \mathscr{L}(\lambda, \theta) = \nabla_\lambda \mathbb{E}_{q(x_0, u_{1:T}, \lambda)} \left[\sum_{t=1}^{T} \log p(y_t \mid x_0, u_{1:t}, \theta) \right] - \nabla_\lambda \mathrm{KL}\big(q(x_0, u_{1:T}, \lambda) \,\|\, p(x_0, u_{1:T})\big)$$

Note that $x_0 \sim \mathcal{N}(\tilde{\mu}_0, \tilde{\Sigma}_0) \iff x_0 = \tilde{\mu}_0 + \tilde{\Sigma}_0^{1/2} \epsilon_0$ where $\epsilon_0 \sim \mathcal{N}(0, I)$.

We can **reparameterize the model** in terms of an expectation wrt $\epsilon_{0:T}$ and then take the gradient inside the expectation, as before

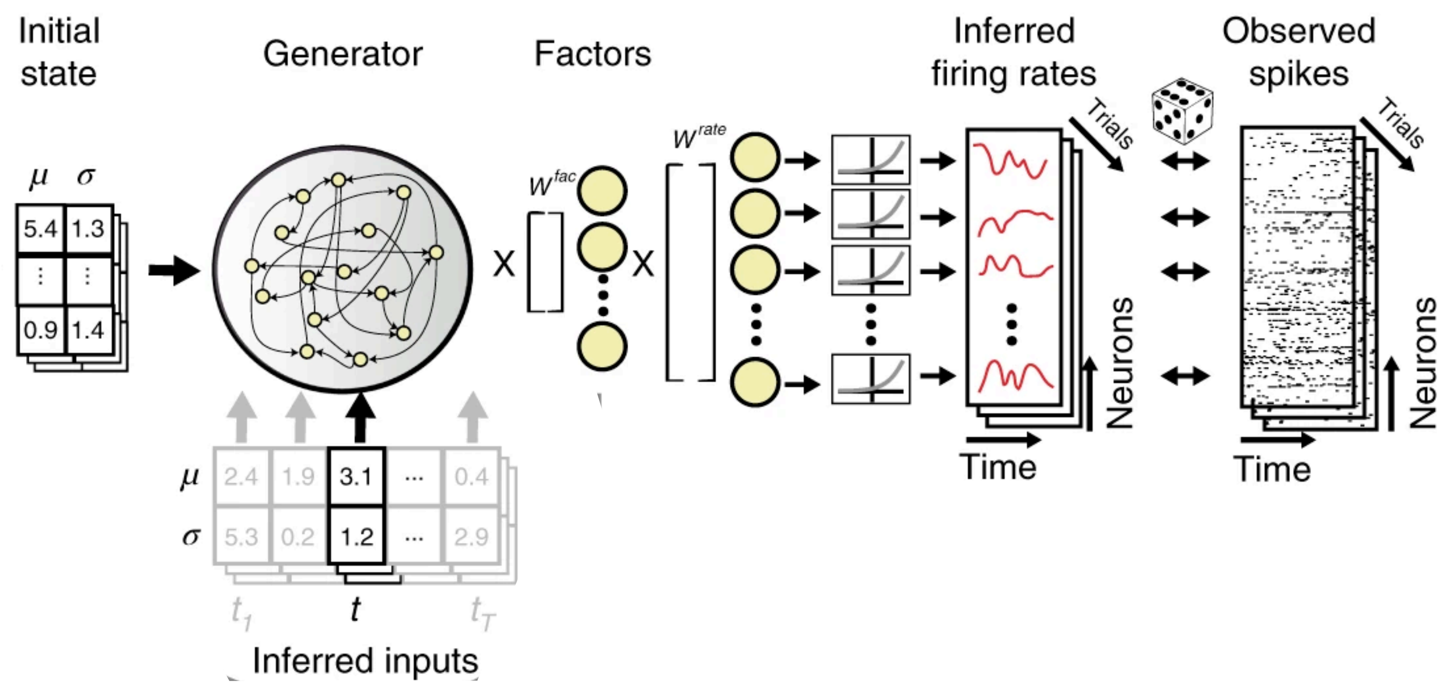
$$\nabla_\lambda \mathscr{L}(\lambda, \theta) = \mathbb{E}_{\epsilon_{0:T}} \left[\sum_{t=1}^{T} \nabla_\lambda \log p(y_t \mid x_0(\epsilon_0, \lambda), u_1(\epsilon_1, \lambda), \ldots, u_t(\epsilon_t, \lambda), \theta) \right] - \nabla_\lambda \mathrm{KL}\big(q(x_0, u_{1:T}, \lambda) \,\|\, p(x_0, u_{1:T})\big)$$

As before, we can approximate this with ordinary Monte Carlo.

Stochastic RNNs

LFADS learning and inference

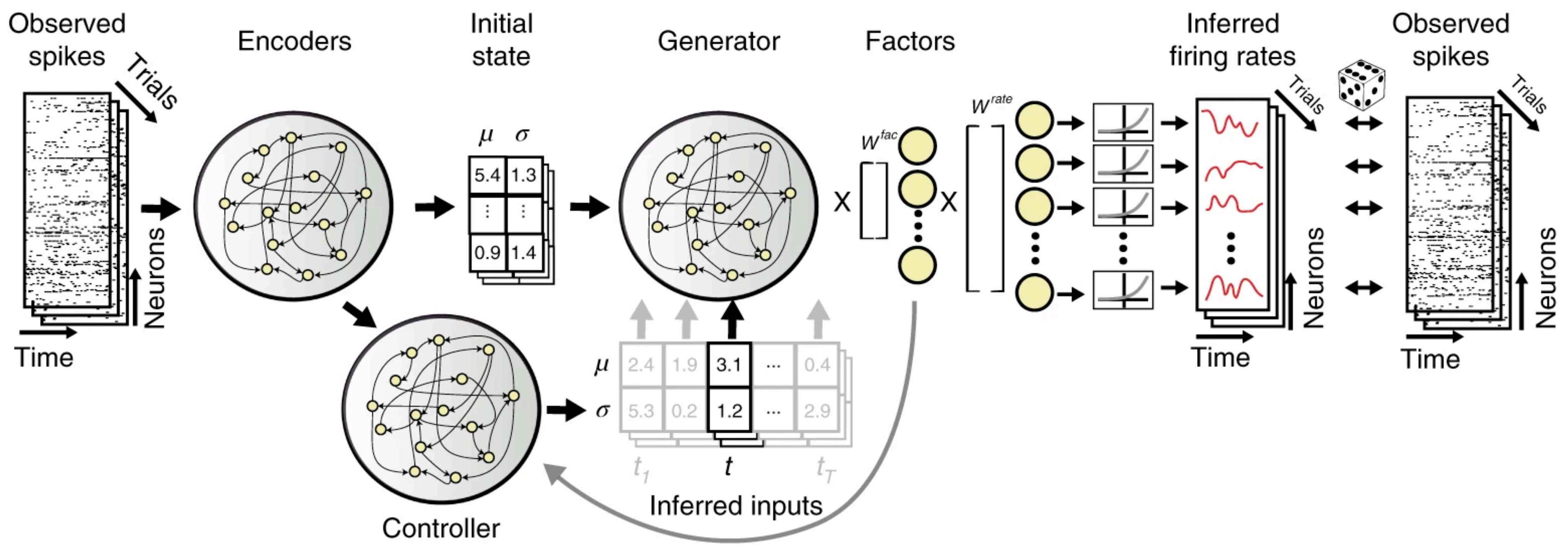
- **Variational EM** via gradient descent and the reparameterization trick,
  - **E step:**
    - Draw $\epsilon_t^{(m)} \sim \mathcal{N}(0,I)$ for $t = 0,\ldots,T$, $s = 1,\ldots,S$.
    - Use ϵ to approximate $\nabla_\lambda \mathscr{L}(\lambda, \theta)$ via Monte Carlo and the reparameterization trick.
    - Update $\lambda \leftarrow \lambda + \alpha \nabla_\lambda \mathscr{L}(\lambda, \theta)$
  - **M step:**
    - Use ϵ to approximate $\nabla_\theta \mathscr{L}(\lambda, \theta)$ via Monte Carlo.
    - Update $\theta \leftarrow \theta + \alpha \nabla_\theta \mathscr{L}(\lambda, \theta)$.



Stochastic RNNs

Amortized inference with encoders / recognition networks

- With large datasets, we often work on one mini-batch at a time.
- In that setting, we need a way to quickly obtain a decent posterior approximation for that mini-batch.
- **Key idea**: the optimal λ is a function of the data $y_{1:T}$, so let's **use a neural network** to approximate the mapping from data to variational parameters.
- This is called **amortized inference.**
- The learned network is called an **encoder** or a **recognition network**.



Conclusion

- **Sequential VAEs** are latent variable models for time series data like neural spike trains and behavioral pose trajectories.
- **LFADS** is one such example that is popular in neuroscience. It uses recurrent neural networks to parameterize the nonlinear dynamics, and Poisson GLMs to model the spike count observations.
- **Learning and inference** are much the same as in standard VAEs —we just maximize the ELBO.
- It also uses an RNN for the **recognition network / encoder**, to estimate latent variables given observations.

Further Reading

- Pandarinath, Chethan, Daniel J. O’Shea, Jasmine Collins, Rafal Jozefowicz, Sergey D. Stavisky, Jonathan C. Kao, Eric M. Trautmann, et al. 2018. “Inferring Single-Trial Neural Population Dynamics Using Sequential Auto-Encoders.” Nature Methods 15 (10): 805–15.