STATS305C: Applied Statistics III

Lecture 16: Poisson processes

Scott Linderman

May 25, 2023

Outline

- Defining properties of a Poisson process
- Four ways to sample a Poisson process
- Beyond Poisson: Doubly stochastic processes

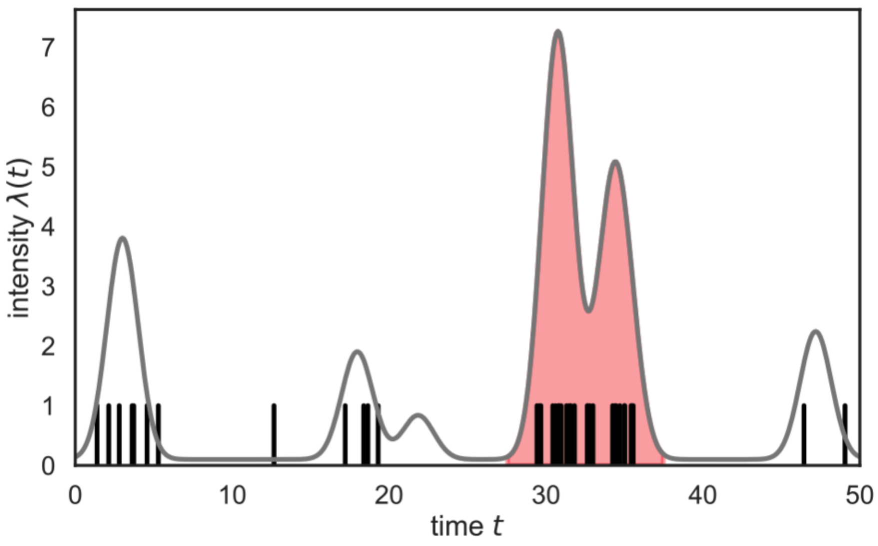
Defining properties of a Poisson process

- Poisson processes are **stochastic processes** that generate **random sets of points** $\{\boldsymbol{x}_n\}_{n=1}^N \subset \mathscr{X}$.
- Poisson processes are governed by an **intensity function**, $\lambda(\boldsymbol{x}) : \mathscr{X} \to \mathbb{R}_+$.
- **Property #1:** The number of points in any interval is a Poisson random variable,

$$N(\mathscr{A}) \sim \mathrm{Po}\left(\int_{\mathscr{A}} \lambda(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\right) \tag{1}$$

- **Property #2:** Disjoint intervals are independent,

$$N(\mathscr{A}) \perp\!\!\!\perp N(\mathscr{B}) \iff \mathscr{A} \cap \mathscr{B} = \varnothing \tag{2}$$



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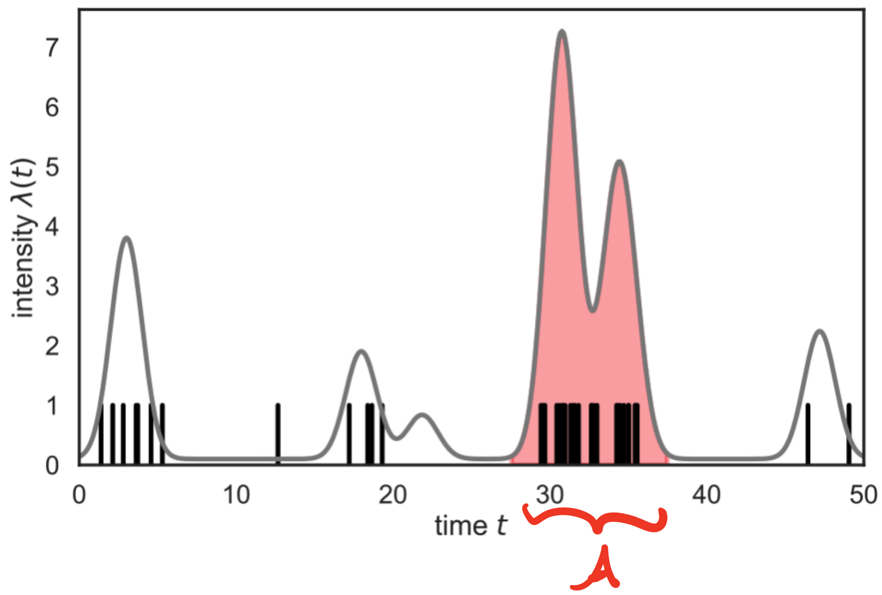

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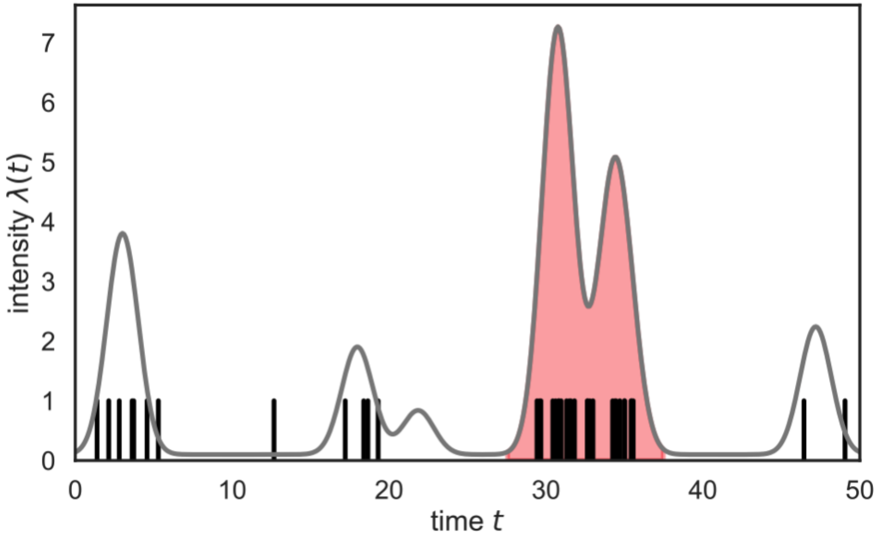


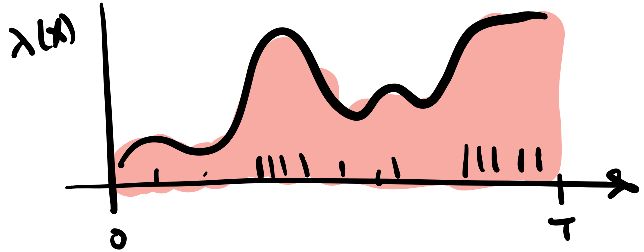
Example applications of Poisson processes

- Modeling neural firing rates
- Locations of trees in a forest
- Locations of stars in astronomical surveys
- Arrival times of customers in a queue (or HTTP requests to a server)
- Locations of bombs in London during World War II
- Times of photon detections on a light sensor
- Others?

Four ways to sample a Poisson process

1. The top-down approach
2. The interval approach
3. The time-rescaling approach
4. The thinning approach

Top-down sampling of a Poisson process



Given $\lambda(\boldsymbol{x})$ (and a domain $\mathscr{X}$):

1. Sample the **total number of points**

$$N \sim \mathrm{Po}\left(\int_{\mathscr{X}} \lambda(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right) \tag{3}$$

2. Sample the **locations** of the points

$$\boldsymbol{x}_n \overset{\text{iid}}{\sim} \frac{\lambda(\boldsymbol{x})}{\int_{\mathscr{X}} \lambda(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'} \Bigg\} \text{pdf} \tag{4}$$

for $n = 1, \ldots, N$.

Question: what assumptions are necessary for this procedure to be tractable?

1. compute integral; 2. Sample iid.

Deriving the Poisson process likelihood

Exercise: from the top-down sampling process, derive the Poisson process likelihood,

$$p\left(\{\boldsymbol{x}_n\}_{n=1}^N \mid \lambda(\boldsymbol{x})\right) = p(N \mid \lambda) \prod_{n=1}^N p(x_n \mid \lambda) \cdot N! \tag{5}$$

$$= \mathrm{Po}\left(N \,\Big|\, \int_{\mathcal{X}} \lambda(x)\,dx\right) \prod_{n=1}^N \frac{\lambda(x_n)}{\int_{\mathcal{X}} \lambda(x)\,dx} \cdot N!$$

$$= \frac{1}{\cancel{N!}}\, e^{-\int_{\mathcal{X}} \lambda(x)\,dx} \cancel{\left[\int_{\mathcal{X}} \lambda(x)\,dx\right]^N} \prod_{n=1}^N \frac{\lambda(x_n)}{\cancel{\int_{\mathcal{X}} \lambda(x)\,dx}} \cdot \cancel{N!}$$

$$\boxed{= e^{-\int_{\mathcal{X}} \lambda(x)\,dx} \prod_{n=1}^N \lambda(x_n)}$$

Intervals of a homogeneous Poisson process

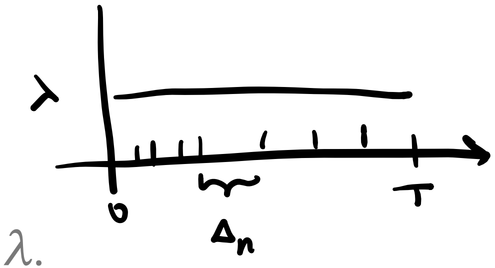
- A Poisson process is **homogeneous** if its intensity is constant, $\lambda(\boldsymbol{x}) \equiv \lambda$.

- **Property #3:** A homogeneous Poisson process on $[0, T] \subset \mathbb{R}$ (e.g. where points correspond to arrival times) has **independent, exponentially distributed intervals**,

$$\Delta_n = x_n - x_{n-1} \overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda) \tag{6}$$

- **Property #4:** A homogeneous Poisson process is **memoryless** — the amount of time until the next point arrives is independent of the time elapsed since the previous point arrived.

Intervals of a homogeneous Poisson process



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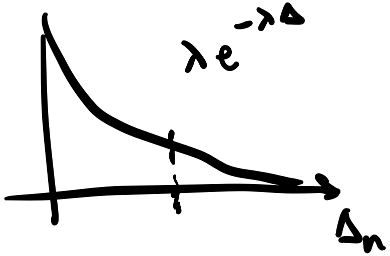

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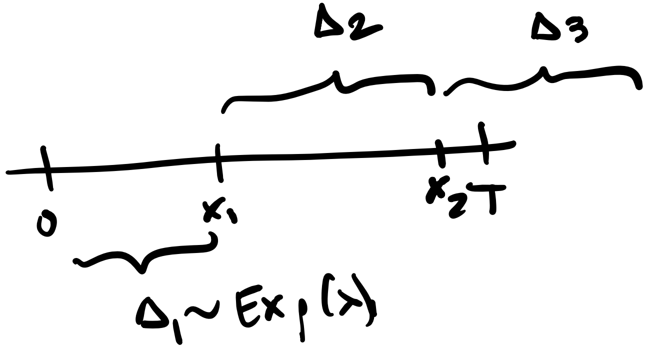
Sampling a homogeneous Poisson process by simulating intervals

We can sample a homogeneous Poisson process on $[0, T]$ by simulating intervals as follows:

1. Initialize $\boldsymbol{X} = \emptyset$ and $x_0 = 0$
2. For $n = 1, 2, \ldots$:
   - Sample $\Delta_n \sim \mathrm{Exp}(\lambda)$.
   - Set $x_n = x_{n-1} + \Delta_n$.
   - If $x_n > T$, break and return $\boldsymbol{X}$,
   - Else, set $\boldsymbol{X} \leftarrow \boldsymbol{X} \cup \{x_n\}$.



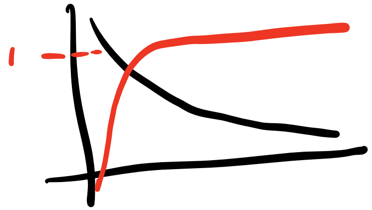
$X = \{x_1, x_2\}$

$x_1 = \Delta_1$

$x_2 = \Delta_1 + \Delta_2$

Deriving the likelihood of a homogeneous Poisson process

Exercise: from the interval sampling process, derive the likelihood of a homogeneous Poisson process. Show that it is the same as what you derived from the top-down sampling process.



$$p(\{x_n\}_{n=1}^{N} \mid \lambda) = p(\{\Delta_n\} \mid \lambda)$$

$$= \left[\prod_{n=1}^{N} \mathrm{Exp}(\Delta_n \mid \lambda)\right] \Pr\left(\Delta_{N+1} > T - \sum_{n=1}^{N} \Delta_n\right)$$

$$= \prod_{n=1}^{N} \lambda e^{-\Delta_n \lambda} \qquad \int_{T-\sum_n \Delta_n}^{\infty} \lambda e^{-\Delta \lambda} \, d\lambda$$

$$= \left[\prod_{n=1}^{N} \lambda e^{-\Delta_n \lambda}\right] e^{-(T - \sum_n \Delta_n)\lambda}$$

$$= \lambda^N e^{-T\lambda} \quad \checkmark$$

$$T\lambda = \int_0^T \lambda \, dx$$

$$\mathcal{X} = [0, T]$$

Sampling an inhomogeneous Poisson process by time-rescaling

- Now consider an **inhomogeneous** Poisson process on $[0, T]$; i.e. one with a non-constant intensity.

- Apply the change of variables,

$$x \mapsto \int_0^x \lambda(t)\,\mathrm{d}t \triangleq \Lambda(x) \tag{7}$$

  Note that this is an **invertible transformation** when $\lambda(x) > 0$.

- Sample a homogeneous Poisson process with unit rate on $[0, \Lambda(T)]$ to get points $\boldsymbol{U} = \{u_n\}_{n=1}^N$. Then set,

$$\boldsymbol{X} = \{\Lambda^{-1}(u_n) : u_n \in \boldsymbol{U}\}. \tag{8}$$

- **Sanity check:** what is the expected value of N?

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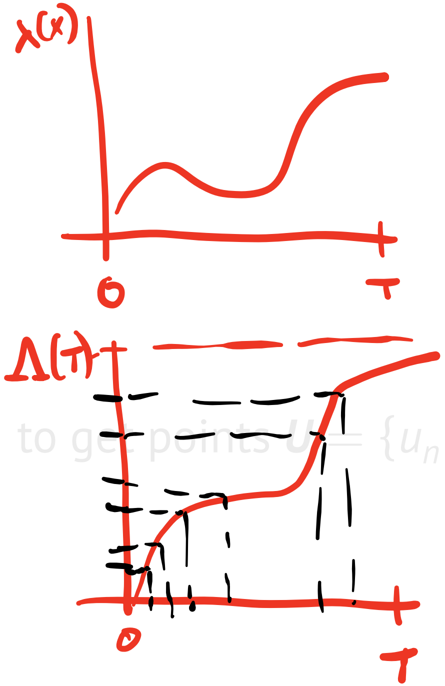

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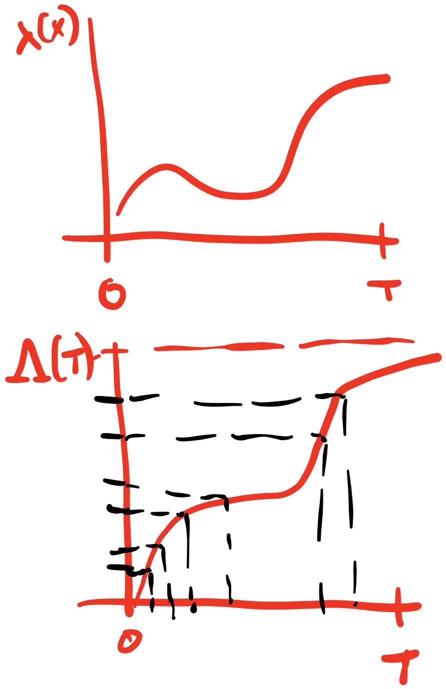
Sampling an inhomogeneous Poisson process by time-rescaling, in pictures

Note: this is the analog of **inverse-CDF** sampling.



A Goodness of fit test for inhomogeneous Poisson processes

- Brown et al. [2002] used the time-rescaling sampling procedure to develop a goodness-of-fit test for inhomogeneous Poisson processes.
- Suppose you observe a set of points $\{x_n\}_{n=1}^N \subset [0,T]$ and you want to test whether they are well-modeled by an inhomogeneous Poisson process with rate $\lambda(x)$.
- Let $\Delta_n = \Lambda(x_n) - \Lambda(x_{n-1})$ with $\Lambda(x_0) = 0$. If the model is a good fit, then $\Delta_n \overset{\text{iid}}{\sim} \mathrm{Exp}(1)$.
- Perform a further transformation $z_n = 1 - e^{-\Delta_n}$. Then $z_n \overset{\text{iid}}{\sim} \mathrm{Unif}([0,1])$.
- Now sort the z_n's in increasing order into $(z_{(1)}, \ldots, z_{(N)})$, so $z_{(1)}$ is the smallest value.
- Intuitively, the points $\left(\frac{n-1/2}{N}, z_{(n)}\right)$ should like along a 45° line.

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A Goodness of fit test for inhomogeneous Poisson processes II

- We can check for significant departures from the 45° line using a simple visual test.
- The order statistics $z_{(n)}$ are marginally beta distributed,

$$z_{(n)} \sim \mathrm{Beta}(n, N-n+1) \tag{9}$$

The mean is $\frac{n}{N+1}$ and its mode is $\frac{n-1}{N-1}$.

- Then, use the 2.5% and 97.5% quantiles of the beta distribution to obtain confidence intervals around the 45° line.

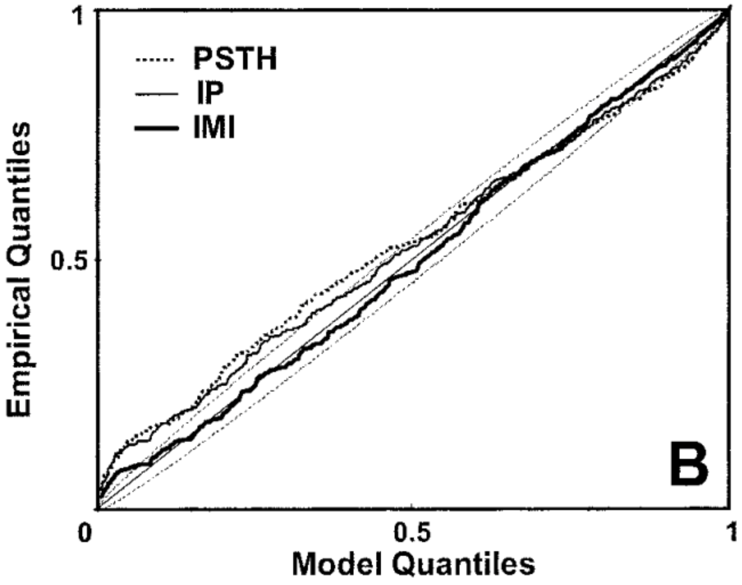

Figure: Figure 1 from Brown et al. [2002].

The Poisson Superposition Principle

- **Property #5:** The union (a.k.a. superposition) of independent Poisson processes is also a Poisson process.

- Suppose we have two independent Poisson processes on the same domain $\mathscr{X}$,

$$\{\boldsymbol{x}_n\}_{n=1}^N \sim \mathrm{PP}(\lambda_1(\boldsymbol{x})) \tag{10}$$

$$\{\boldsymbol{x}'_m\}_{m=1}^M \sim \mathrm{PP}(\lambda_2(\boldsymbol{x})) \tag{11}$$

  Then

$$\{\boldsymbol{x}_n\}_{n=1}^N \cup \{\boldsymbol{x}'_m\}_{m=1}^M \sim \mathrm{PP}(\lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})) \tag{12}$$

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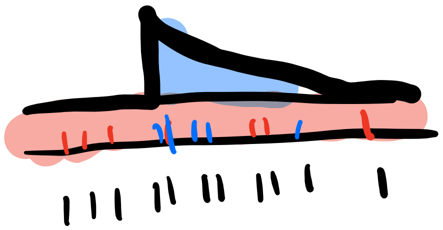

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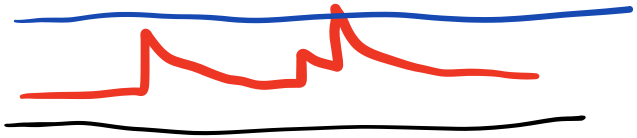
Poisson thinning



- The opposite of Poisson superposition is **Poisson thinning**.
- Suppose we have points $\{\boldsymbol{x}_n\}_{n=1}^N \sim \mathrm{PP}(\lambda(\boldsymbol{x}))$ where $\lambda(\boldsymbol{x}) = \lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})$.
- Sample independent binary variables

$$z_n \sim \mathrm{Bern}\left(\frac{\lambda_1(\boldsymbol{x}_n)}{\lambda_1(\boldsymbol{x}_n) + \lambda_2(\boldsymbol{x}_n)}\right). \tag{13}$$

- Then $\{\boldsymbol{x}_n : z_n = 1\} \sim \mathrm{PP}(\lambda_1(\boldsymbol{x}))$ and $\{\boldsymbol{x}_n : z_n = 0\} \sim \mathrm{PP}(\lambda_2(\boldsymbol{x}))$.

eg. if $\lambda(x) \equiv \lambda$

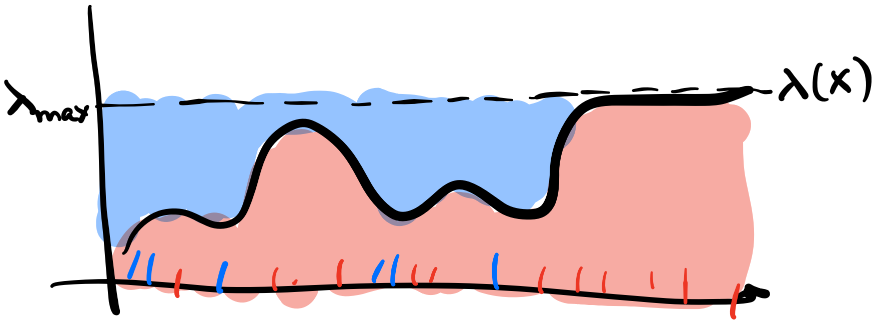
$N \sim \mathrm{Po}(\lambda T)$

$x_n \overset{iid}{\sim} \mathrm{Unif}([0,T])$

Sampling a Poisson process by thinning

Exercise: Use Poisson thinning to sample an inhomogeneous Poisson process with a bounded intensity, $\lambda(\boldsymbol{x}) \leq \lambda_{\max}$.

Question: What Monte Carlo sampling method is this akin to?



$M \sim \mathrm{Po}(\lambda_{max} \cdot T)$

$z_m \sim \mathrm{Bern}\left(\frac{\lambda(x_m)}{\lambda_{max}}\right)$

Outline

- Defining properties of a Poisson process
- Four ways to sample a Poisson process
- **Beyond Poisson**

What's not to love about Poisson processes?

- independence

Conditional intensity functions

- One way of introducing dependence is via an **autoregressive model**. Consider a point process on a time interval $[0, T]$.
- Let $\lambda(t \mid \mathscr{H}_t)$ denote a **conditional intensity function** where $\mathscr{H}_t$ is the **history** of points before time t.
- Technically, $\mathscr{H}_t$ is a **filtration** in the language of stochastic processes.
- Allowing past points to influence the intensity function enables more complex, non-Poisson models.

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Hawkes processes

- Hawkes processes [Hawkes, 1971] are **self-exciting point processes**.

- Their conditional intensity function is modeled as,

$$\lambda(t \mid \mathcal{H}_t) = \lambda_0 + \sum_{t_n \in \mathcal{H}_t} h(t - t_n), \tag{14}$$

  where $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a positive **impulse response** or **influence function**.

- For example, the impulse responses could be modeled as exponential functions,

$$h(\Delta t) = \tfrac{w}{\tau} e^{-\frac{\Delta t}{\tau}} = w \cdot \mathrm{Exp}(\Delta t; \tau), \tag{15}$$

  where $\tau \in \mathbb{R}_+$ is a time-constant governing the rate of decay and $w \in \mathbb{R}_+$ is a scaling parameter such that $\int_0^\infty h(\Delta t)\, \mathrm{d}\Delta t = w$.

Hawkes processes

- Hawkes processes [Hawkes, 1971] are **self-exciting point processes**.

- Their conditional intensity function is modeled as,

$$\lambda(t \mid \mathcal{H}_t) = \lambda_0 + \sum_{t_n \in \mathcal{H}_t} h(t - t_n), \tag{14}$$

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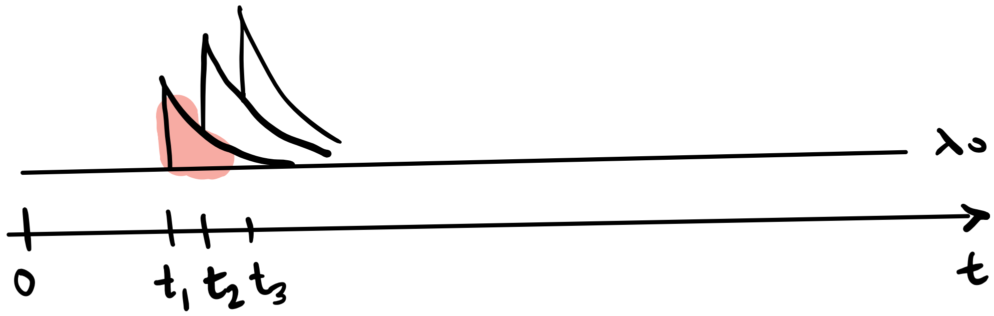
Hawkes processes, in pictures

$h(t - t_n) = w \cdot \mathrm{Exp}(t - t_n; \tau)$



\# self-exciting

Maximum likelihood estimation for Hawkes processes I

- Suppose we observe a collection of time points $\{t_n\}_{n=1}^N \subset [0, T]$ and want to estimate the parameters $\boldsymbol{\theta} = (\lambda_0, w)$ of a Hawkes process with an exponential impulse response function. (Consider τ to be fixed.)

- The Hawkes process log likelihood is just like that of a Poisson process,

$$\log p(\{t_n\}_{n=1}^N \mid \boldsymbol{\theta}) = -\int_0^T \lambda_{\boldsymbol{\theta}}(t \mid \mathcal{H}_t) \, \mathrm{d}t + \sum_{n=1}^N \log \lambda_{\boldsymbol{\theta}}(t_n \mid \mathcal{H}_t) \tag{16}$$

Maximum likelihood estimation for Hawkes processes II

- Substituting in the form of the conditional intensity, we can simplify the log likelihood to,

$$\log p(\{t_n\}_{n=1}^N \mid \boldsymbol{\theta}) = -\int_0^T \Big[\lambda_0 + w \sum_{t_n \in \mathcal{H}_t} \mathrm{Exp}(t - t_n; \tau)\,\mathrm{d}t\Big] + \sum_{n=1}^N \log\Big(\lambda_0 + w \sum_{t_m \in \mathcal{H}_{t_n}} \mathrm{Exp}(t_n - t_m; \tau)\Big) \tag{17}$$

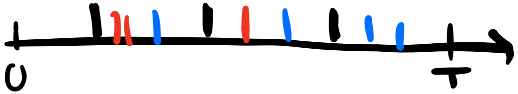
$$\approx -\boldsymbol{\theta}^\top \boldsymbol{\phi}_0 + \sum_{n=1}^N \log\Big(\boldsymbol{\theta}^\top \boldsymbol{\phi}_n\Big) \tag{18}$$

where $\boldsymbol{\phi}_0 = (T, N)^\top$ and $\boldsymbol{\phi}_n = \Big(1, \sum_{t_m \in \mathcal{H}_{t_n}} \mathrm{Exp}(t_n - t_m; \tau)\Big)^\top$.

- **Questions:** What approximation did we make? How would you maximize the log likelihood as a function of $\boldsymbol{\theta}$?

Marked point processes

- Now suppose we observed points from S difference **sources**.

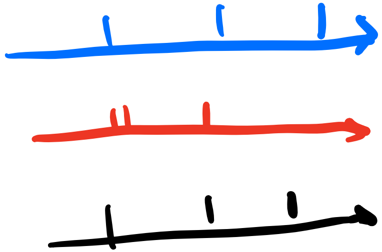


- We can represent the points as a set of tuples, $\{(t_n, s_n)\}_{n=1}^N$ where $t_n \in [0, T]$ denotes the time and $s_n \in \{1, \ldots, S\}$ denotes the source of the n-th point.

- We will model them as a **marked point process**.

- Like before, we have a (conditional) intensity function, but this time is defined over time and marks,

$$\lambda(t, s \mid \mathscr{H}_t) : [0, T] \times \{1, \ldots, S\} \mapsto \mathbb{R}_+ \tag{19}$$



- When s takes on a discrete set of values, we often use the shorthand,

$$\lambda_s(t \mid \mathscr{H}_t) \triangleq \lambda(t, s \mid \mathscr{H}_t) \tag{20}$$

to denote the intensity for the s-th source.

Multivariate Hawkes processes

- A **multivariate** Hawkes process is a marked point process with **mutually excitatory** interactions.

- It is defined by the conditional intensity functions,

$$\lambda_s(t \mid \mathscr{H}_t) = \lambda_{s,0} + \sum_{(t_n,s_n)\in\mathscr{H}_t} h_{s_n,s}(t - t_n). \tag{21}$$

  where $h_{s,s'}(\Delta t)$ is a **directed impulse response** from points on source s to the intensity of s'.

- Again, it is common to model the impulse responses as weighted probability densities; e.g.,

$$h_{s,s'}(\Delta t) = w_{s,s'} \cdot \mathrm{Exp}(\Delta t; \tau_{s,s'}) \tag{22}$$

  where $w_{s,s'}$ is the weight.

- Like before, the weights can be estimated using maximum likelihood estimation.

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Multivariate Hawkes Processes II

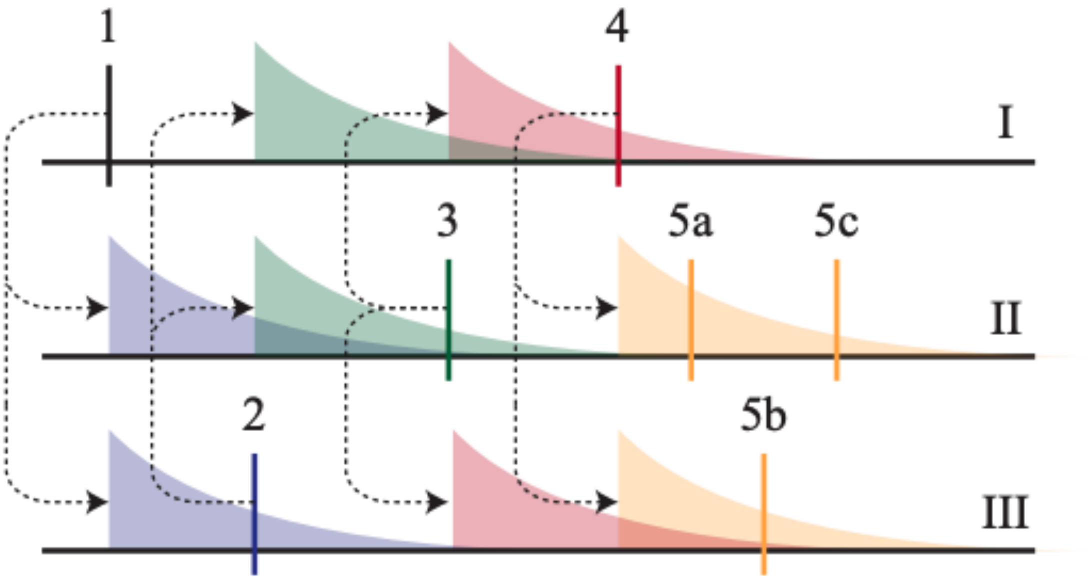

Figure 1: Illustration of a Hawkes process. Events induce impulse responses on connected processes and spawn “child” events. See the main text for a complete description.

From Linderman and Adams [2014].

Discovering latent network structure in point process data

- We can think of the weights as defining a **directed network**,

$$\boldsymbol{W} = \begin{bmatrix} w_{1,1} & \cdots & w_{1,S} \\ \vdots & & \vdots \\ w_{S,1} & \cdots & w_{S,S} \end{bmatrix} \tag{23}$$

  where $w_{s,s'} \in \mathbb{R}_+$ is the strength of influence that events (points) on source s induce on the intensity of source s'.

- However, we don't directly observe the network. We only observed it indirectly through the point process.

- **Question:** when is a multivariate Hawkes process stable, in that the intensity tends to a finite value in the infinite time limit?

Multivariate Hawkes processes as Poisson clustering processes

- Note that the conditional intensity in eq. (21) is a sum of a background intensity and a bunch of non-negative impulse responses.

$$\lambda_s(t \mid \mathscr{H}_t) = \lambda_{0,s} + \sum_{(t_n, s_n) \in \mathscr{H}_t} h_{s_n,s}(t - t_n). \tag{24}$$

- **Question:** which property of Poisson processes applied to such intensities?

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$$\lambda_s(t \mid \mathcal{H}_t) = \lambda_{s,0} + \sum_{(t_n,s_n)\in\mathcal{H}_t} h_{s_n,s}(t - t_n). \tag{25}$$

- **Question:** which property of Poisson processes applied to such intensities?

- Using the **Poisson superposition principle**, we can partition the points $\mathcal{T}_s = \{t_n : s_n = s\}$ from source s into **clusters** attributed to either the background or to one of the impulse responses.

$$\mathcal{T}_s = \bigcup_{n=0}^{N} \mathcal{T}_{s,n} \tag{26}$$

where

$$\mathcal{T}_{s,0} \sim \mathrm{PP}(\lambda_{s,0}) \qquad \text{[background points]} \tag{27}$$

$$\mathcal{T}_{s,n} \sim \mathrm{PP}(h_{s_n,s}(t - t_n)) \qquad \text{[points induced by } (t_n, s_n)\text{]} \tag{28}$$

Multivariate Hawkes processes as Poisson clustering processes

- Now the weights have an intuitive interpretation. Plugging in the definition of the impulse response,

$$\mathcal{T}_{s,n} \sim \mathrm{PP}\Big(w_{s_n,s} \cdot \mathrm{Exp}(t - t_n; \tau_{s_n,s})\Big). \tag{29}$$

- **Question:** What is the expected number of points induced by this impulse response, i.e. $\mathbb{E}[|\mathcal{T}_{s,n}|]$?

Conjugate Bayesian inference for multivariate Hawkes processes

- Let's put a gamma prior on the weights,

$$w_{s,s'} \sim \mathrm{Ga}(\alpha, \beta). \tag{30}$$

- **Question:** suppose we know the partition of points; i.e. we knew the clusters $\mathcal{T}_{s,n}$. What is the conditional distribution,

$$p\left(w_{s,s'} \mid \{\{\mathcal{T}_{s,n}\}_{n=0}^{N}\}_{s=1}^{S}\right) = \tag{31}$$

Conjugate Bayesian inference for multivariate Hawkes processes II

- We don't know the partition of spikes in general, but we do know its conditional distribution!
- Let $z_n \in \{0, \ldots, n-1\}$ denote the cluster to which the n-th spike is assigned, with $z_n = 0$ denoting the background cluster. With this notation,

$$\mathcal{T}_{s,n} = \{(t_{n'}, s_{n'}) : s_{n'} = s \wedge z_{n'} = n\}. \tag{32}$$

- **Question:** what is the conditional distribution of the cluster assignment,

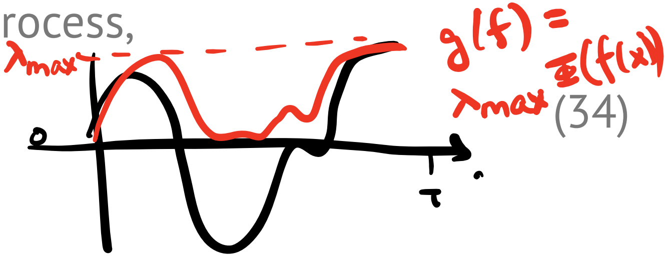
$$p(z_n \mid \{(t_n, s_n)\}_{n=1}^{N}; \boldsymbol{\theta}) = \tag{33}$$

- Using these two conditional distributions, we can derive a simple Gibbs sampling algorithm that alternates between sampling the weights given the clusters and the clusters given the weights.

Beyond Poisson: Doubly stochastic processes

- Hawkes processes are only one way of going beyond Poisson processes.
- Whereas Hawkes processes take an autoregressive approach, **doubly stochastic point processes** (a.k.a. **Cox processes**) take a latent variable approach.
- In these models, the intensity itself is modeled as a stochastic process,

$$\lambda(\boldsymbol{x}) \sim p(\lambda). \tag{34}$$



- For example, consider the model,

$$\lambda(\boldsymbol{x}) = g(f(\boldsymbol{x})) \quad \text{where} \quad f \sim \mathrm{GP}(\mu(\cdot), K(\cdot, \cdot)). \tag{35}$$

  When g is the exponential function, this is called a **log Gaussian Cox process**. When g is the sigmoid function, this is called a **sigmoidal Gaussian Cox process** [Adams et al., 2009].

- Aternatively, take λ to be a convolution of a Poisson process with a non-negative kernel; this is called a Neyman-Scott process [Wang et al., 2022, e.g.].

$$f \sim GP(\mu, K)$$

$$\lambda(x) = \lambda_{max}\, \Phi(f(x))$$

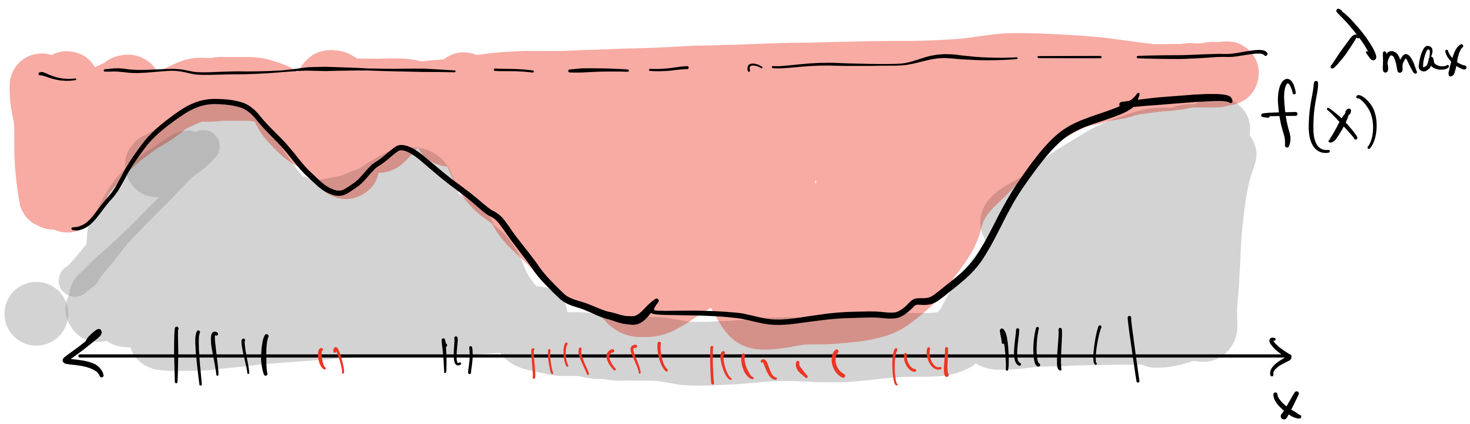
$$\{x_n\}_{n=1}^{N} \sim PP(\lambda(x))$$

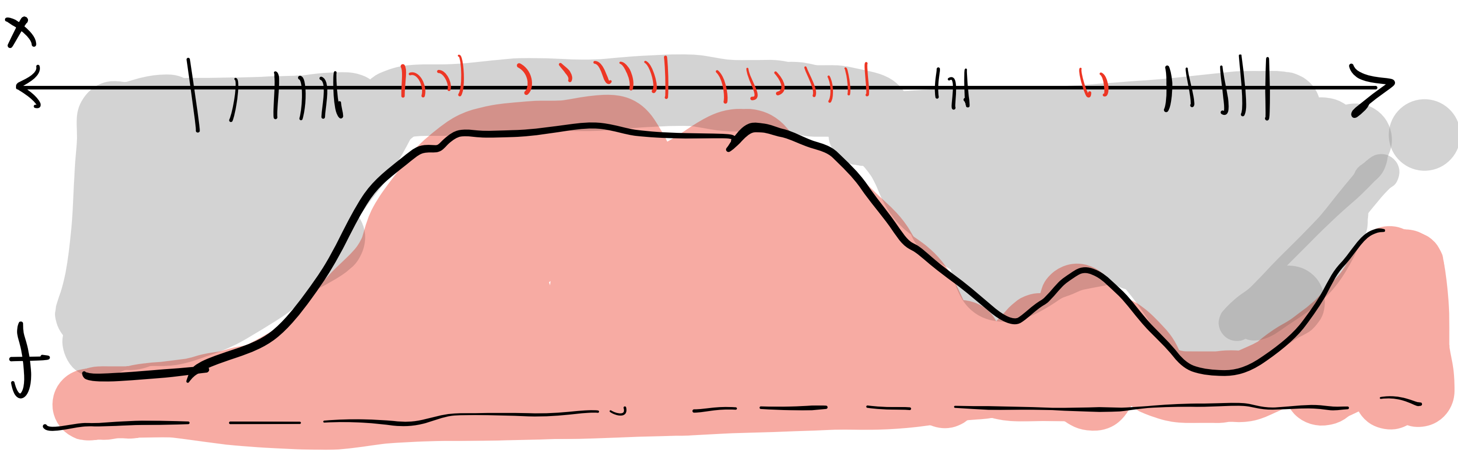
$$\{r_m\}_{m=1}^{M} \sim PP(\lambda_{max})$$

$$y_m \sim \mathrm{Bern}\left(\frac{\lambda(r_m)}{\lambda_{max}}\right)$$

$$= \mathrm{Bern}(\Phi(f(r_m)))$$

$$\text{Set } \{x_n\}_{n=1}^{N} = \{r_m : y_m = 1\}$$


λmax
f(x)
x


x
f

Conclusion

- Poisson processes are stochastic processes that generate discrete sets of points.
- They are defined by an intensity function $\lambda(\boldsymbol{x})$, which specifies the expected number of points in each interval of time or space.
- We can build in dependencies by conditioning on past points or introducing latent variables.
- Poisson process modeling boils down to inferring the intensity. We can take various parametric and nonparametric approaches.
- The hardness comes about when the integral in the Poisson process likelihood is intractable.
- As we will see next time, Poisson processes are also mathematical building blocks for Bayesian nonparametric models with random measures.

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